ON A SERIES OF PRODUCTS OF THREE GEGENBAUER POLYNOMIALS*

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In a note in the Proceedings of the American Mathematical Society, John P. Vinti has established the following theorem:

THEOREM.—If \( x, y, z \) are real variables and \( P_n \) denotes the Legendre polynomial of order \( n \), then

\[
\sum_{n=0}^{\infty} (n + \frac{1}{2}) P_n(x) P_n(y) P_n(z) = \begin{cases} 
\pi^{-1} g^{-\frac{1}{2}} & (g > 0) \\
0 & (g < 0) \\
1 < x, y, z < +1 
\end{cases}
\]

where

\[
g = g(x, y, z) = 1 - x^2 - y^2 - z^2 + 2xyz
\]

The object of the present note is to prove a similar relation involving the Gegenbauer (ultraspherical) polynomials.

Let \( x, y, z \) be real variables and \( C_n^\nu \) the Gegenbauer polynomial of order \( n \). We prove

\[
\sum_{n=0}^{\infty} \left( \frac{n!}{\Gamma(n + 2\nu)} \right)^2 (n + \nu) C_n^\nu(x) C_n^\nu(y) C_n^\nu(z) = \begin{cases} 
\frac{4^{1-2\nu}}{\pi g^{\nu-1}} & (g > 0) \\
0 & (g < 0) \\
-1 < x, y, z < +1; \nu > 0
\end{cases}
\]

As in [1], if we denote by \( T_+ \) and \( T_- \) the regions as bounded above, i.e., \(-1 < x, y, z < +1\), wherein \( g > 0 \) and \( g < 0 \) respectively, the left-hand side of (3) converges uniformly with respect to \( x \) or \( y \) or \( z \) alone in any closed interval (parallel to the \( x \) or \( y \) or \( z \) axis) interior to \( T_+ \) or \( T_- \).

*After completing the work of this paper, I have learnt that Dr. Brafman (Wayne University Detroit, U.S.A.) has communicated a paper on the topic.
Introduce the function

\[
f(x, y, z) = \begin{cases} \frac{4^{1-2\nu} \pi}{(|\nu|^4)} \frac{g^{\nu-1}}{(1-x^2)(1-y^2)(1-z^2)^{\nu-1}} & (g > 0) \\ 0 & (g \leq 0) \end{cases} \]

\[ -1 \leq x, y, z \leq 1 ; \quad \nu > 0 \]  

From the expansion

\[
f(x, y, z) = \sum_{n=0}^{\infty} A_n f_n (y, z) C_n^\nu (x) \] (5)

we have after a formal calculation

\[
\int_{-1}^{1} f(x, y, z) (1 - x^2)^{\nu-1} C_n^\nu (x) \, dx = A_n f_n (y, z) \frac{2^{1-2\nu} \pi n!}{(n + \nu) (|\nu|^2)^{\nu}} . \] (6)

As shown in[1], \( g > 0 \) if and only if

\[ x_1 = yz - \sqrt{(1-y^2)(1-z^2)} < x < x_2 = yz + \sqrt{(1-y^2)(1-z^2)} , \]

so that the integral on the left-hand side of (6) can be written as

\[
\frac{4^{1-2\nu} \pi}{(|\nu|^4)} \int_{x_1}^{x_2} \frac{g^{\nu-1}}{(1-y^2)(1-z^2)^{\nu-1}} C_n^\nu (x) \, dx . \]

By the substitution

\[ x = yz + \sqrt{(1-y^2)(1-z^2)} \cos \phi , \]

the above reduces to

\[
\frac{4^{1-2\nu} \pi}{(|\nu|^4)} \int_{0}^{\pi} C_n^\nu \{ yz + \sqrt{(1-y^2)(1-z^2)} \cos \phi \} (\sin \phi)^{2\nu-1} \, d\phi . \]

Using the addition formula for Gegenbauer polynomials,[2] the expression is seen to be

\[
\frac{2^{1-2\nu} n! \pi}{(|\nu|^2)^{\nu}} C_n^\nu (y) C_n^\nu (z) . \]

From (6) we have then

\[ A_n f_n (y, z) - \left( \frac{n!}{n+2\nu} \right)^2 (n + \nu) C_n^\nu (y) C_n^\nu (z) . \] (7)

Comparing (5) and (7), we have

\[
f(x, y, z) = \sum_{n=0}^{\infty} \left( \frac{n!}{n+2\nu} \right)^2 (n + \nu) C_n^\nu (x) C_n^\nu (y) C_n^\nu (z) . \] (8)

It remains to examine the validity of the expansion (5). As \( f(x, y, z) \) is piecewise continuous in \(-1 \leq x, y, z \leq 1\), we observe[3] that if the integrals

\[
l_1 = \int_{-1}^{+1} |f(x, y, z)| \, dx, \quad l_2 = \int_{-1}^{+1} (1 - x^2)^{\nu-1/2} |f(x, y, z)| \, dx \]

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exist, the series expansion (5) is valid in the interior of \((- 1, + 1)\) and the convergence is uniform in every closed interval interior to \((- 1, + 1)\). Also the expansion (5) is valid at the end points \(x = \pm 1\) if \(\nu < 0\).

Now we show that both \(I_1\) and \(I_2\) exist if \(\nu > 0\). For

\[
I_1 = \frac{4^{1-2\nu} \pi}{(|\nu|)^4} \left\{ (1 - y^2) (1 - z^2) \right\}^{-(\nu - \frac{1}{2})} \int_{z_1} \left\{ (1 - x^2)^{-\nu/2} g^{\nu-1} \right\} dx,
\]

and with the substitution

\[
x = yz + \sqrt{(1 - y^2) (1 - z^2)} \cos \phi
\]

\[
I_1 = \frac{4^{1-2\nu} \pi}{(|\nu|)^4} \int_0 \left( \sin \phi \right)^{2\nu-1} d\phi.
\]

The last integral exists if \(\nu > 0\).

\[
I_2 = \frac{4^{1-2\nu} \pi}{(|\nu|)^4} \left\{ (1 - y^2) (1 - z^2) \right\}^{-(\nu - \frac{1}{2})} \int_{z_1} (1 - x^2)^{-\nu/2} g^{\nu-1} dx
\]

exists if \(x_1^2, x_2^2 \neq 1\). If \(x_1^2 = x_2^2 = 1\), we have \(y = -z\) or \(y = z\) and in both these cases we can write

\[
I_2 = \frac{4^{1-2\nu} (2\pi)}{(|\nu|)^4} \cdot \frac{|a|^{3\nu/2 - 1}}{2^{\nu/2}} \int_0^{\pi/2} \left( \cos \theta \right)^{\nu-1} (\sin \theta)^{2\nu-1} (2 - a^2 \cos^2 \theta)^{-\nu/2} d\theta
\]

where

\[
0 < a^2 = 2 (1 - y^2) \leq 2.
\]

As

\[
(2 - a^2 \cos^2 \theta)^{-\nu/2} \leq (2 \sin^2 \theta)^{-\nu/2} \text{ for } \nu > 0.
\]

\[
I_2 \leq \frac{4^{1-2\nu} \pi}{(|\nu|)^4} \cdot \frac{a^{3\nu/2 - 1}}{2^{\nu/2}} \cdot 2^{-\nu/2} B \left( \frac{\nu}{2}, \frac{\nu}{2} \right)
\]

which certainly exists for \(\nu > 0\). If \(y^2 = z^2 = 1\), \(I_2\) reduces to 0.

At the end points \(x = \pm 1\), the expansion (5) is not valid as that requires \(\nu < 0\). For the same reason we conclude that (8) is not valid at the points \(y = \pm 1\) and \(z = \pm 1\).

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REFERENCES