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Short Communication

2 X 2 Matrix-multiplication revisited

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Abstract

In this note, an algorithm for 2 $\times$ 2 matrix-multiplication is described, and an application of this is made to 3 $\times$ 3 matrix-multiplication.

Key words: Matrix-multiplication, computer algorithm.

1. Introduction

Strassen's algorithm[1] for multiplying two 2 $\times$ 2 matrices, with entries from an arbitrary ring R, involves 7 multiplications and 18 additions (assuming that addition and subtraction are the same kind of operations). Subsequently, Winograd[2] discovered a more efficient algorithm, involving 7 multiplications but only 15 additions. In this note, we give an alternative algorithm, which also involves 7 multiplications and 15 additions, and a combination of the two algorithms is applied to 3 $\times$ 3 matrix-multiplication.

2. Algorithm

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

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be matrices, where $a_{ij}, b_{ij} \in R$, $1 \leqslant i, j \leqslant 2$. Then, the alternative algorithm is given by the identities below:

$$a_{11}b_{11} + a_{12}b_{21} = t + (a_{12} - a_{22})(b \quad + b_{21}) + (a_{12} - a_{11})(b_{21} - b_{22})$$
$$a_{11}b_{12} + a_{12}b_{22} = t + a_{11}[(b_{12} - b_{11}) - (b_{21} - b_{22})] + (a_{12} - a_{22})(b_{11} + b_{21})$$
$$a_{21}b_{11} + a_{22}b_{21} = t + b_{11}[(a_{21} - a_{11}) + (a_{12} - a_{22})] + (a_{12} - a_{11})(b_{21} - b_{22})$$

where $t = a_{22}b_{22} + (a_{11} - (a_{12} - a_{22}))(b_{11} + (b_{21} - b_{22}))$. The term $a_{21}b_{12} + a_{22}b_{22}$ is computed as it is. If intermediate results are appropriately saved, it is easy to see that the algorithm requires 7 multiplications and 15 additions.

3. Application

To compute the product of the matrices, $P = [p_{ij}]$ and $Q = [q_{ij}]$, $p_{ij}, q_{ij} \in R$, $1 \leqslant j \leqslant 3$, we have to compute the terms $\sum_{j=1}^{3} p_{ij} q_{jk}$, $1 \leqslant i, k \leqslant 3$. This can be done in 25 multiplications, by combining Winograd's scheme with ours.

We first note that in Winograd's algorithm the term $a_{11}b_{11} + a_{12}b_{21}$ of the product AB is computed as it is.

The partial sums $\sum_{j=1}^{2} p_{ij} q_{jk}$, $1 \leqslant i, k \leqslant 2$, can be computed by multiplying the matrices

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \text{ and } \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix},$$

according to the above algorithm in 7 multiplications. The partial sums $\sum_{j=2}^{3} p_{ij} q_{jk}$, $2 \leqslant i, k \leqslant 3$, can be computed by multiplying the matrices

$$\begin{bmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{bmatrix} \text{ and } \begin{bmatrix} q_{22} & q_{23} \\ q_{32} & 2_{33} \end{bmatrix}$$

by Winograd's algorithm in 6 more multiplications, since $p_{22}\, q_{22}$ is available from the first step; 12 more multiplications are needed to compute all the terms of PQ, and this brings the tally to 25. This result was found by Gastinel[3] in a more involved way.

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References

1. AHO, A., HOPCROFT, J. E., AND ULLMAN, J. D. *The design and analysis of computer algorithms*, Addison-Wesley Reading, MA., 1974, pp. 230–231.

2. KNUTH, D. E. *The art of computer programming*, Vol. II, Addison-Wesley, Reading MA, 1981, pp. 481–482.

3. GASTINEL M. Sur le calcul des produits de matrices, *Numer. Math.*, 1971, 222–229.