Non-symmetrical bending of circular plates having small initial curvature under the combined action of lateral loads and forces in the middle plane of the plates

DEBABRATA SINHA* AND SUKUMAR BASULI
Department of Mathematics, Tripura Engineering College, Tripura

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Abstract

The note aims at solving the problem of bending of a circular plate in a non-symmetrical case where the plate had an initially given small deflection, and the plate bending subjected to the combined action of lateral loads and forces in the middle plane of the plate.

Key words: Circular plates, lateral loads and forces, non-symmetrical bending

1. Introduction

Several problems on the bending of circular plates (symmetrical as well as non-symmetrical cases) under the combined action of lateral loads and forces in the middle plane of the plate have been discussed by Timoshenko and Woinowsky-Krieger. However we believe that the problems of non-symmetrical bending of circular plates having small initial curvature and under the action of lateral loads and forces have not been studied.

The object of this paper is to solve the problem of non-symmetrical bending of circular plates having a small initial curvature under the action of lateral loads and forces.

2. Problem

The general differential equation in polar co-ordinates for the bending of plates under the action of lateral loads and forces in the middle plane of the plate is given by

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) = \frac{q}{D} - \frac{N}{D} \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right)
\]

(1.0)

* Udaipur Kirit Bikram Institution, P.O. Radhakishorepur, Tripura State, India.
where $q =$ Lateral load, $N =$ Uniform compressive force acting in the middle plane of the plate and $D =$ Flexural rigidity of the plate.

If $w_0$ denotes the initial small deflection and $w_1$ is the additional deflection as a result of the applied load and force, we can write for a small total deflection.

$$w = w_0 + w_1.$$ (1.1)

Now since the l.h.s. of equation (1.0) was obtained from the expressions for the bending moments in the plate which were dependent only on the change in curvature, it follows that $W$ on the l.h.s. of (1.0) must be replaced by $W_1$ only. On the other hand, the effect of applied lateral load and force being dependent on the total curvature, $W$ on the r.h.s. of equation (1.0) must be replaced by $W_0 + W_1$ in this problem. Thus for the initially curved circular plate equation (1.0) becomes:

$$
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)
= \frac{q}{D} - \frac{N}{D} \left[ \frac{\partial^2}{\partial r^2} (W_0 + W_1) + \frac{1}{r} \frac{\partial}{\partial r} (W_0 + W_1) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} (W_0 + W_1) \right]$$

(1.2)

3. Solution of the problem

Let us now try to solve the equation (1.2) in the case where the initial curvature is defined by the relation:

$$W_o = \sum_{m=0}^{\infty} A_{om} (a^2 - r^2)^2 \cos m\theta + \sum_{m=1}^{\infty} B_{om} (a^2 - r^2)^2 \sin m\theta.$$ (1.3)

Let the load be of the form

$$q = \sum_{m=0}^{\infty} f_m (r) \cos m\theta + \sum_{m=1}^{\infty} F_m (r) \sin m\theta$$ (1.4)

where $f_m (r)$ and $F_m (r)$ are functions of $r$ only.

Substituting (1.3) and (1.4) in (1.2) we obtain

$$
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)
+ \frac{N}{D} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)
= \frac{1}{D} \left[ \sum_{m=0}^{\infty} f_m (r) \cos m\theta + \sum_{m=1}^{\infty} F_m (r) \sin m\theta \right]
$$
\[ + \frac{N}{D} \sum_{m=0}^{\infty} A_{om} \left\{ (16 - m^2) r^2 + (2m^2 - 8) a^2 - \frac{m^2 \alpha^4}{r^2} \right\} \cos m\theta \]

\[ + \sum_{m=0}^{\infty} B_{om} \left\{ (16 - m^2) r^2 + (2m^2 - 8) a^2 - \frac{m^2 \alpha^4}{r^2} \right\} \sin m\theta \]  \quad (1.5)

If we now assume for \( W_1 \) the form
\[ W_1 = R_0 + \sum_{m=1}^{\infty} R_m \cos m\theta + \sum_{m=1}^{\infty} R'_m \sin m\theta \]  \quad (1.6)

where \( R_0, R_m, R'_m \) are functions of \( r \) only, we get by (1.5) and (1.6) the following differential equations for \( R_m, R'_m \) and \( R_0 \),
\[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} \right) \left( \frac{d^2 R_m}{dr^2} + \frac{1}{r} \frac{dR_m}{dr} - \frac{m^2 R_m}{r^2} \right) \]
\[ + \frac{N}{D} \left( \frac{d^2 R_m}{dr^2} + \frac{1}{r} \frac{dR_m}{dr} - \frac{m^2 R_m}{r^2} \right) \]
\[ = \frac{1}{D} f_m(r) + \frac{N}{D} A_{om} \left\{ (16 - m^2) r^2 + (2m^2 - 8) a^2 - \frac{m^2 \alpha^4}{r^2} \right\} \]  \quad (1.7)

\[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} \right) \left( \frac{d^2 R'_m}{dr^2} + \frac{1}{r} \frac{dR'_m}{dr} - \frac{m^2 R'_m}{r^2} \right) \]
\[ + \frac{N}{D} \left( \frac{d^2 R'_m}{dr^2} + \frac{1}{r} \frac{dR'_m}{dr} - \frac{m^2 R'_m}{r^2} \right) \]
\[ = \frac{1}{D} f'_m(r) + \frac{N}{D} B_{om} \left\{ (16 - m^2) r^2 + (2m^2 - 8) a^2 - \frac{m^2 \alpha^4}{r^2} \right\} \]  \quad (1.8)

\[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \left( \frac{d^2 R_0}{dr^2} + \frac{1}{r} \frac{dR_0}{dr} \right) + \frac{N}{D} \left( \frac{d^2 R_0}{dr^2} + \frac{1}{r} \frac{dR_0}{dr} \right) \]
\[ = f_0(r) - \frac{8NA_{oo}}{D} (\alpha^2 - 2r^2). \]  \quad (1.9)

To find the solution of (1.7) let us put
\[ \frac{d^2 R_m}{dr^2} + \frac{1}{r} \frac{dR_m}{dr} - \frac{m^2 R_m}{r^2} + \frac{N}{D} R_m = \chi_m(r) \]

Then we get from (1.7), the following equation:
\[
\frac{r^2}{D} \frac{d^2}{dr^2} \chi_m + r \frac{d}{dr} \chi_m - m^2 \chi_m = \frac{r^2}{D} f_m(r) + \frac{NA_{om}}{D} \left\{ (16 - m^2) r^4 + (2m^2 - 8) a^2 - \frac{m^2 \alpha^4}{r^2} \right\}. \]  \quad (2.0)
The solution for $\chi_m$ is:
\[
\chi_m (r) = C_m r^m + D_m r^{-m} + \psi_m (r) + \frac{NA_{om}}{D} (r^4 - 2a^2 r^2 + a^4)
\]
where $\psi_m (r) = \text{Particular integral due to the term } \frac{r^2 f_m (r)}{D} \text{ in (2.0)}$. Replacing $\chi_m$ by this expression we get:
\[
\frac{d^2 R_m}{dr^2} + \frac{1}{r} \frac{dR_m}{dr} + \left( \frac{N}{D} - \frac{m^2}{r^2} \right) R_m
= C_m r^m + D_m r^{-m} + \psi_m (r) + \frac{N}{D} A_{om} (r^4 - 2a^2 r^2 + a^4).
\tag{2.1}
\]
Now putting
\[
\frac{N}{D} = \frac{1}{l^2}, \quad r = lz, \quad \frac{d}{dr} = \frac{1}{l} \frac{dz}{dz},
\]
equation (2.1) takes the form:
\[
Z^2 \frac{d^2 R_m}{dz^2} + z \frac{dR_m}{dz} + (z^2 - m^2) R_m
= C_m (lz)^{m+2} + D_m (lz)^{-m+2} + l^2 z^2 \psi_m (lz) + A_{om} (l^4 z^6 - 2a^2 l^2 z^4 + a^4 z^2).
\tag{2.2}
\]
Now let
\[
R_m = A_m J_m (z) + B_m y_m (z)
\tag{2.3}
\]
be the general solution of (2.2). Then by the method of variation of parameters we have
\[
A_m' J_m (z) + B_m' y_m (z) = 0
A_m' J_m' (z) + B_m' y_m' (z)
= C_m (lz)^{m+2} + D_m (lz)^{-m+2} + l^2 z^2 \psi_m (lz) + A_{om} (l^4 z^6 - 2a^2 l^2 z^4 + a^4 z^2)
\]
Solving for $A_m'$ and $B_m'$, we have
\[
A_m' = \left[ C_m (lz)^{m+2} + D_m (lz)^{-m+2} + l^2 z^2 \psi_m (lz) + A_{om} (l^4 z^6 - 2a^2 l^2 z^4 + a^4 z^2) \right] y_m (z)
\]
\[
J_m' (z) y_m (z) - J_m (z) y_m' (z)
\tag{2.4}
\]
\[
B_m' = \left[ C_m (lz)^{m+2} + D_m (lz)^{-m+2} + l^2 z^2 \psi_m (lz) + A_{om} (l^4 z^6 - 2a^2 l^2 z^4 + a^4 z^2) \right] J_m (z)
\]
\[
J_m (z) y_m' (z) - J_m' (z) y_m (z)
\tag{2.5}
\]
Integration of (2.4) and (2.5) gives:

\[ A_m = -C_m \int l^{m+2} z^{m+3} y_m(z) \, dz - D_m l^{-m+2} \int z^{-m+3} y_m(z) \, dz \]
\[ - l^2 \int z^3 \psi_m(lz) y_m(z) \, dz + A_{om} [ - l^4 \int z^7 y_m(z) \, dz ] + E_m \]
\[ + 2a^2 l^2 \int z^5 y_m(z) \, dz - a^4 \int z^3 y_m(z) \, dz + E_m \]  
(2.6)

\[ B_m = C_m l^{m+2} \int z^{m+3} J_m(z) \, dz + D_m l^{-m+2} \int z^{-m+3} J_m(z) \, dz \]
\[ + l^2 \int z^3 \psi_m(lz) J_m(z) \, dz + A_{om} [ l^4 \int z^7 J_m(z) \, dz ] \]
\[ - 2a^2 l^2 \int z^5 J_m(z) \, dz + a^4 \int z^3 J_m(z) \, dz + F_m \]  
(2.7)

So the general solution for \( R_m \) is of the form:

\[ R_m = C_m l^{m+2} \phi_{m+3, m}(z) + D_m l^{-m+2} \phi_{-m+3, m}(z) + l^2 \theta_m(z) \]
\[ + A_{om} [ l^4 \phi_{7, m}(z) - 2a^2 l^2 \phi_{5, m}(z) + a^4 \phi_{3, m}(z) ] \]
\[ + E_m J_m(z) + F_m y_m(z) \]  
(2.8)

where

\[ \phi_{n, m}(z) = y_m(z) \int z^n J_m(z) \, dz - J_m(z) \int z^n y_m(z) \, dz \]
\[ = y_m(z) \left[ \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{z}{2} \right)^{m+2k}}{k! \sqrt{k + m + 1} \cdot (n + m + 2k + 1)} \right] \]  
\[ + \frac{\pi}{2} J_m(z) \left[ \sum_{k=0}^{m-1} \frac{(-1)^k \left( \frac{z}{2} \right)^{m+2k}}{k! \cdot (m + n + 2k + 1)} \right] \]
\[ + z^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{z}{2} \right)^{m+2k}}{k! \cdot m + k \cdot (m + n + 2k + 1)^2} \]
\[ - z^n \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{z}{2} \right)^{m+2k}}{k! \cdot m + k \cdot (m + n + 2k + 1)} \]
\[ \times \left\{ \log \frac{z}{2} - \frac{1}{2} \psi(k + 1) - \frac{1}{2} \psi(m + k + 1) \right\} \]  
(2.9)
in which
\[
\psi (m + 1) = \sum_{k=1}^{\infty} k^{-1} + \psi (1)
\]
\[
\psi (1) = -0.57722
\]
for \( n = m + 3, -m + 3, 3, 5 \) and 7.

Also
\[
\theta_m (z) = \psi_m (l) J_m (z) \int z^3 \psi_m (l) J_m (z) \, dz - J_m (z) \int z^3 \psi_m (l) \psi_m (z) \, dz.
\]
similar expressions can be obtained for \( R_m' \) by solving the equation (1.8).

By solving the equation (1.9), the expression for \( R_n \) can be obtained in the form (Sinha).\(^4\)

\[
R_0 = \sum_{n=1}^{\infty} \left[ \frac{2 \int_0^R r J_0 (\alpha_n r)}{\alpha^2 n (\alpha^2_n - k^2) J_0^2 (\alpha_n a)} + \frac{64 k^2 A_0 J_0 (\alpha_n a)}{\alpha^2_n (\alpha^2_n - k^2) J_0^2 (\alpha_n a)} \right] \times [J_0 (\alpha_n r) - J_0 (\alpha_n a)]
\]
in which
\[
k^2 = \frac{N}{D}
\]
and \( \alpha_n \) is the \( n \)-th root of the equation
\[
J_1 (\alpha a) = 0.
\]

Substituting these expressions for \( R_0, R_m \) and \( R_m' \) in (1.6) we obtain the additional deflection \( W_1 \). With this value of \( W_1 \) and taking \( W_0 \) from (1.3) we can get the total deflection \( W \) at any point \((r, \theta)\) of the circular plate.

The constants of integration \( C_m, D_m, E_m \) and \( F_m \) in each particular case must be determined so as to satisfy the given boundary conditions.

A particular case: Let us consider the case of a circular plate with free boundary. Such a condition is found in the case of a circular foundation slab, supporting a chimney. We assume here that the moment \( M \) produced due to the wind pressure produces reactions in the slab following a linear law and thus we obtain similar type of loading as discussed in the foregoing.
The boundary conditions at the outer boundary of the plate, which is free, are

\[ [M_r]_{r=a} = 0 \]  \hspace{1cm} (3.3)

\[ [V_r]_{r=a} = \left( Q_r - \frac{1}{r} \frac{\partial}{\partial r} M_r \right)_{r=a} = 0. \]  \hspace{1cm} (3.4)

We also assume here that the inner portion of the plate of radius \( b \) is considered rigid and the edge of the plate along the circle of radius \( b \) is clamped. For this the following boundary conditions must be satisfied.

\[ \left[ \frac{\partial W}{\partial r} \right]_{r=b} = \left[ \frac{W}{r} \right]_{r=b} \]  \hspace{1cm} (3.5)

Let the load be of the form

\[ f_1 (r) = q = q_{10} \cdot r \]  \hspace{1cm} (3.6)

As the solution of the equation (1.2) we take only the term of the series (1.6) that contains the function \( R_1 \). We assume

\[ W = \left[ A_0 \right] \left( a^2 - r^2 \right)^2 + c_1 l^3 \phi_{4,1} \left( \frac{r}{l} \right) + D_1 l \phi_{2,1} \left( \frac{r}{l} \right) + \frac{q_{10} l^5}{8 D} \phi_{6,1} \left( \frac{r}{l} \right) \]

\[ + A_0 \left\{ l^4 \phi_{7,1} \left( \frac{r}{l} \right) - 2a^2 l^2 \phi_{5,1} \left( \frac{r}{l} \right) + a^4 \phi_{3,1} \left( \frac{r}{l} \right) \right\} \]

\[ + E_1 J_1 \left( \frac{r}{l} \right) + F_1 y_1 \left( \frac{r}{l} \right) \] \hspace{1cm} (3.7)

where

\[ l^2 \theta_m (z) = \frac{q_{10} l^5}{8 D} \phi_{6,1} \left( \frac{r}{l} \right) \]

and

\[ \phi_{n,1} \left( \frac{r}{l} \right) = \frac{1}{l^{n+1}} y_1 \left( \frac{r}{l} \right) \int r^n J_1 \left( \frac{r}{l} \right) dr \]

\[ - \frac{1}{l^{n+1}} J_1 \left( \frac{r}{l} \right) \int r^n y_1 \left( \frac{r}{l} \right) dr \]  \hspace{1cm} (3.8)

for \( n = 2, 3, 4, 5, 6 \) and 7 only.

Let us now take \( a = 10 \) units, \( b = 1 \) unit, \( l = 10 \) units. Then for \( m = 1 \), we take \( F_1 = 0 \) so as to eliminate an infinitely large bending moment at the central portion of the plate.
Solving the equations obtained by using the boundary conditions, we obtain the values of the Constants $C_1$, $D_1$ and $E_1$:

\[
C_1 = 17.537 \frac{q_{10}}{D} + 3541.2313 A_{01}
\]

\[
D_1 = -3113.207 \frac{q_{10}}{D} + 277645.6 A_{01}
\]

\[
E_1 = 0.
\]

Thus in this particular example, we have from (3.5), the following expression for the deflection:

\[
W = \left[ A_{01} \left(10^2 - r^2\right)^2 + \left(17.537 \frac{q_{10}}{D} + 3541.2313 A_{01}\right) 10^3 \phi_{4,1} \left(\frac{r}{10}\right) + \left(277645.6 A_{01} - 3113.207 \frac{q_{10}}{D}\right) \phi_{2,1} \left(\frac{r}{10}\right) + 10^5 \cdot \frac{q_{10}}{D} \phi_{6,1} \left(\frac{r}{10}\right) + A_{01} 10^4 \left\{ \phi_{2,1} \left(\frac{r}{10}\right) - 2\phi_{5,1} \left(\frac{r}{10}\right) + \phi_{3,1} \left(\frac{r}{10}\right) \right\} \right] \cos \theta.
\]

We can now tabulate the deflection in any direction, say for $\theta = 60^\circ$ and for the different values of the distances from the centre of the plate.

### Table I

**Deflections at different points**

Here we have chosen $A_{01} = \frac{q_{10}}{100 D}$.

<table>
<thead>
<tr>
<th>Deflection ((W))</th>
<th>(r = 2) Units</th>
<th>(r = 4) Units</th>
<th>(r = 6) Units</th>
<th>(r = 8) Units</th>
<th>(r = 10) Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( WD ) (\frac{100q_{10}}{})</td>
<td>0.4676</td>
<td>1.034</td>
<td>2.375</td>
<td>3.670</td>
<td>5.178</td>
</tr>
</tbody>
</table>

**References**

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2. **Timoshenko and Woinowsky-Krieger**

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4. **Sinha, D.**