ELECTROMAGNETIC BOUNDARY VALUE PROBLEM OF THE DIELECTRIC-COATED CONDUCTING SPHERE EXCITED BY DELTA-FUNCTION ELECTRIC AND MAGNETIC SOURCES

(MRS.) R. CHATTERJEE

(Department of Electrical Communication Engineering, Indian Institute of Science
Bangalore 560012)

Received on July 12, 1973 and in revised form on January 23, 1974

ABSTRACT

The electromagnetic boundary-value problem of the dielectric-coated conducting sphere excited by delta-function electric and magnetic sources applied normally across an arbitrary plane, has been solved. The possibility of symmetric as well as unsymmetric TE, TM and hybrid modes have been investigated.

Key words: Electromagnetic scattering, Dielectric coated conducting sphere, Boundary value problem.

1. INTRODUCTION

Scattering of electromagnetic waves from conducting, dielectric and dielectric-coated conducting spheres have been studied by several authors [1-7]. The problem of forced oscillations of a conducting sphere which is excited in an infinite number of symmetric TM modes by a delta-function electric source field applied normally across the equatorial plane, has been considered by Stratton and Chu [8] in 1941. In an earlier paper [9] the present author has studied the electromagnetic boundary-value problem of the dielectric sphere excited by delta-function electric and magnetic sources applied normally across an arbitrary plane, and discussed the possibility of exciting TM, TE and hybrid modes. The present author with others has also studied the problem of radiation from a dielectric-coated metal spherical antenna excited in the unsymmetric hybrid mode [10].

In this paper, the electromagnetic boundary-value problem of a dielectric-coated conducting sphere excited by delta function electric and magnetic sources applied normally across an arbitrary plane has been solved. The possibility of exciting both symmetric and unsymmetric TM, TE and hybrid modes have been investigated.

I.I.Sc.- 1
2. STATEMENT OF THE PROBLEM

The geometry of the structure is given in Fig. 1. Spherical co-ordinates \( r, \theta, \phi \) are used. A perfectly conducting sphere of radius \( 'a' \) and constants \( \epsilon_0, \mu_0, \sigma = \infty \), coated by a dielectric of constants \( \epsilon_1, \mu_1, \sigma_1 \) and of thickness \( 'b-a' \), is embedded in another dielectric medium of constants \( \epsilon_0, \mu_0, \sigma_0 \). The dielectric-coated conducting sphere is excited by delta-function electric and magnetic field sources in a direction normal to the plane \( z = z_1 = b \cos \theta_1 \).

The object of this paper is to solve the electromagnetic boundary-value problem and to discuss the possibility of the existence of hybrid, \( TM \) and \( TE \) modes.

3. HYBRID MODES

Let the excitation of the dielectric-coated conducting sphere be a combination of an electric field \( E' e^{-j\omega t} \) and a magnetic field \( H' e^{-j\omega t} \) applied uniformly over the plane \( z = z_1 = b \cos \theta_1 \), and in a direction normal to this plane.

Let
\[
E' = E_0 \cos m\phi
\]
\[
H' = H_0 \cos m\phi
\]

Both \( E' \) and \( H' \) have components \( E_r, E_\theta, H_r, H_\theta \) in the \( r \) and \( \theta \) directions respectively. These components of the applied electric and magnetic fields can be expanded in series of spherical harmonies as follows:

\[
E_r' (r, \theta, \phi)
= -\frac{n(n+1)}{k_1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_{mn} (r) \cos (m\phi) P_n^m (\cos \theta) e^{-j\omega t}
\]

\[
\sin \theta E_\theta' (r, \theta, \phi)
= -\frac{1}{k_1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} (r) \cos (m\phi) P_n^m (\cos \theta) e^{-j\omega t}
\]

\[
H_r' (r, \theta, \phi)
= -\frac{n(n+1)}{j\omega \mu_1} \sum_{m=0}^{\infty} D_{mn} (r) \sin (m\phi) P_n^m (\cos \theta) e^{-j\omega t}
\]

\[
\sin \theta H_\theta' (r, \theta, \phi)
= -\frac{1}{\mu_1} \sum_{m=0}^{\infty} C_{mn} (r) \sin (m\phi) P_n^m (\cos \theta) e^{-j\omega t}
\]
Dielectric-Coated Conducting Sphere

\[ = \frac{\mu_0}{\mu_1 \mu_0} N'_{m, m+1} h_{m+1}^{(1)} (k_0 b) \]  
\[ \frac{1}{\mu_1} \left\{ L_m, m+1 j_{m+1} (k_1 a) + M_m, m+1 y_{m+1} (k_1 a) \right\} = 0. \]

Equations (48)-(59) are twelve equations in the twelve unknown amplitude coefficients \( L_{mm}, M_{mm}, N_{mm}, L'_{mm}, M'_{mm}, N'_{mm}, L_{m, m+1}, N_{m, m+1}, M_{m, m+1}, L'_{m, m+1}, M'_{m, m+1}, N'_{m, m+1} \). Hence these unknown coefficients can be solved for.

Putting \( n = m + 1, m + 2, m + 3, \ldots \), etc., in equations (39)-(47), the other higher order coefficients \( L_m, m+2, M_m, m+2, N_m, m+2, L'_{m, m+2}, M'_{m, m+2}, N'_{m, m+2}, \ldots \), etc., can be solved for. Since equations (39)-(47) contain the amplitude coefficients \( L_{mn}, M_{mn}, N_{mn}, L'_{mn}, M'_{mn}, N'_{mn}, L_{m, n-1}, M_{m, n-1}, N_{m, n-1}, L'_{m, n-1}, M'_{m, n-1}, N'_{m, n-1}, L_{m, n+1}, M_{m, n+1}, N_{m, n+1}, L'_{m, n+1}, M'_{m, n+1}, N'_{m, n+1}, \) it is not possible to separate out only the coefficients \( L_{mn}, M_{mn}, N_{mn}, L'_{mn}, M'_{mn}, N'_{mn} \) for the same value of \( n \).

Hence the boundary conditions are satisfied not for a single value of \( n \) but for \( n - 1, n, \) and \( n + 1 \) combined together. This shows that for a hybrid mode, for any particular value of \( m \), the electric and magnetic field components consist of an infinite number of terms for values of \( n \) varying from \( m \) to \( \infty \).

Since for each value of \( m \), the infinite number of amplitude coefficients \( L_{mn}, L'_{mn}, M_{mn}, M'_{mn}, N_{mn}, N'_{mn} \) can be solved for \( n = m \) to \( \infty \), it can be concluded that for each value of \( m \), there exists a corresponding hybrid mode.

4. **Unsymmetric TM and TE Modes**

For an unsymmetric TM mode \( (m \neq 0) \), let the excitation be \( E' e^{-jwt} \) applied uniformly over the plane \( z = z_1 = a \cos \theta_1 \), and in a direction normal to this plane.

Let

\[ E' = E_0 \cos (m\phi) \]  

\( E' \) has components \( E'_r \) and \( E'_\theta \) in the \( r \) and \( \theta \) directions. These components \( E'_r \) and \( E'_\theta \) can be expanded in series of spherical harmonics as given by equations (3), (4), (7), (9), (13) and (15).
The field components inside the dielectric coating are:

\[
E_r^i = - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n (n + 1) \cos (m\phi) P_n^m (\cos \theta) \left\{ L_{mn} j_n (k_1 r) + M_{mn} y_n (k_1 r) \right\} e^{-j\omega t} + E_r' \quad (61)
\]

\[
E_\theta^i = - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{d}{d\theta} \{ P_n^m (\cos \theta) \} \cos (m\phi) \frac{1}{k_1 r} \left\{ L_{mn} j_n (k_1 r) \right\}' + M_{mn} [k_1 r y_n (k_1 r)]' e^{-j\omega t} + E_\theta' \quad (62)
\]

\[
E_\phi^i = \frac{1}{\sin \theta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_n^m (\cos \theta) m \sin (m\phi) \frac{1}{k_1 r} \left\{ L_{mn} [k_1 r j_n (k_1 r)]' + M_{mn} [k_1 r y_n (k_1 r)] \right\}' e^{-j\omega t} \quad (63)
\]

\[
H_r^i = 0 \quad (64)
\]

\[
H_\theta^i = \frac{k_1}{j\omega \mu_1 \sin \theta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_n^m (\cos \theta) m \sin (m\phi) \left\{ L_{mn} j_n (k_1 r) + M_{mn} y_n (k_1 r) \right\} e^{-j\omega t} \quad (65)
\]

\[
H_\phi^i = \frac{k_1}{j\omega \mu_1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{d}{d\theta} \{ P_n^m (\cos \theta) \} \cos (m\phi) \left\{ L_{mn} j_n (k_1 r) + M_{mn} y_n (k_1 r) \right\} e^{-j\omega t} \quad (66)
\]

The field components outside the dielectric coating are

\[
E_r^e = - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n (n + 1) N_{mn} \cos (m\phi) P_n^m (\cos \theta) \left\{ \sum_{n_1}^{(1)} \frac{h_{n_1}(k_\theta r)}{k_\theta r} \right\} e^{-j\omega t} \quad (67)
\]

\[
E_\theta^e = - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{d}{d\theta} \{ P_n^m (\cos \theta) \} N_{mn} \cos (m\phi) \frac{1}{k_\theta r} \left\{ \sum_{n_1}^{(1)} [k_\theta r h_{n_1}(k_\theta r)] \right\}' e^{-j\omega t} \quad (68)
\]

\[
E_\phi^e = \frac{1}{\sin \theta} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} N_{mn} P_n^m (\cos \theta) m \sin (m\phi) \frac{1}{k_\theta r} \left\{ \sum_{n_1}^{(1)} [k_\theta r h_{n_1}(k_\theta r)] \right\}' e^{-j\omega t} \quad (69)
\]
Dielectric-Coated Conducting Sphere

\[ H_r^e = 0 \]  
\[ H_\theta^e = j\omega \mu_0 \sin \theta \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m N_{mn} P_n^m (\cos \theta) \sin (m\phi) h_n^{(1)} (k_0 r) e^{-j\omega t} \]  
\[ H_\phi^e = \frac{k_0}{j\omega \mu_0} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} N_{mn} \frac{d}{d\theta} \left\{ P_n^m (\cos \theta) \cos (m\phi) h_n^{(1)} (k_0 r) e^{-j\omega t} \right\} \]

Applying the boundary conditions that

(i) \( E_\theta^i = E_\theta^e, \ E_\phi^i = E_\phi^e, \ H_\theta^i = H_\theta^e, \ H_\phi^i = H_\phi^e \) at \( r = b \)

and

(ii) \( E_\theta^i = 0, \ E_\phi^i = 0 \) at \( r = a \)

we obtain

\[ \frac{1}{k_1 b} \left\{ L_{mn} [k_1 b j_n (k_1 b)]' + M_{mn} [k_1 b y_n (k_1 b)]' \right\} + \frac{C_{mn} (b)}{k_1} = \frac{N_{mn}}{k_0 b} \left\{ k_0 b h_n^{(1)} (k_0 b) \right\}' \]  
\[ \frac{1}{k_1} \left\{ L_{mn} [k_1 b j_n (k_1 b)]' + M_{mn} [k_1 b y_n (k_1 b)]' \right\} = \frac{N_{mn}}{k_0} \left\{ k_0 b h_n^{(1)} (k_0 b) \right\}' \]  
\[ \frac{k_1}{\mu_1} \left\{ L_{mn} j_n (k_1 b) + M_{mn} y_n (k_1 b) \right\} = \frac{k_0}{\mu_0} N_{mn} h_n^{(1)} (k_0 b) \]  
\[ \frac{k_1}{\mu_1} \left\{ L_{mn} j_n (k_1 b) + M_{mn} y_n (k_1 b) \right\} = \frac{k_0}{\mu_0} N_{mn} h_n^{(1)} (k_0 b) \]

Equations (75) and (76) are the same and equations (77) and (78) are the same. Equations (73), (74), (75) and (77) are four equations in the three unknowns \( L_{mn}, M_{mn}, \) and \( N_{mn}. \) Hence there is no unique solution for \( L_{mn}, M_{mn} \) and \( N_{mn}. \)

This shows that forced unsymmetric TM modes are not possible on the dielectric coated metal sphere.
It can be shown in a similar manner that forced unsymmetric TE modes are also not possible on the dielectric coated metal sphere.

5. SYMMETRIC TM AND TE MODES

For a symmetric TM mode \((m = 0)\), let the excitation be \(E' e^{-i\omega t}\) applied uniformly over the plane \(z = z_1 = a\cos \theta_1\), and in a direction normal to this plane. Let \(E' = E_0\), and \(E'\) have components.

\[
E_r' = E_0 \cos \theta_1 = E_{\theta_0} \quad \text{and} \quad E_\theta' = E_{\theta_0} = -E_0 \sin \theta,
\]

in the \(r\) and \(\theta\) directions. These components \(E_r'\) and \(E_\theta'\) can be expanded in series of spherical harmonies as given below:

\[
E_{\theta_0} = -\frac{1}{k_1} \sum_{n=0}^{\infty} C_{\theta n}(r) P_{n+1}(\cos \theta) e^{-i\omega t}
\]

\[
E_{r_0} = -\frac{1}{k_1} \sum_{n=0}^{\infty} n(n+1) D_{\theta n}(r) P_n(\cos \theta) e^{-i\omega t}
\]

where

\[
C_{\theta n}(r) = -\frac{k_1(2n+1)}{2\pi} \int_0^{2\pi} \int_0^\pi E_{\theta n}(r) P_{n+1}(\cos \theta) \sin \theta \, d\theta d\phi
\]

\[
D_{\theta n}(r) = -\frac{k_1(2n+1)}{n(n+1)2\pi} \int_0^{2\pi} \int_0^\pi E_{r n}(r) P_n(\cos \theta) \sin \theta \, d\theta d\phi.
\]

If \(E_0\) is a \(\delta\)-function source given by equations (11) and (12), then

\[
C_{\theta n}(r) = C_{\theta n}(a) = C_{\theta n}(b)
\]

\[
= -\frac{V}{b} \frac{k_1(2n+1)}{2n(n+1)} \sin \theta_1 P_{n+1}(\cos \theta_1)
\]

\[
D_{\theta n}(r) = D_{\theta n}(a) = D_{\theta n}(b)
\]

\[
= \frac{V}{b} \cos \theta_1 P_n(\cos \theta_1) \frac{k_1(2n+1)}{2n(n+1)}
\]

The field components inside the sphere are

\[
E_{r} = -\sum_{n=0}^{\infty} n(n+1) \frac{k_1}{k_1 r} P_n(\cos \theta) (L_{\theta n}j_n(k_1 r) + M_{\theta n}y_n(k_1 r)) e^{-i\omega t} + E_{\theta_0}
\]
\[ E_{\theta}^i = - \sum_{n=0}^{\infty} \frac{d}{d\theta} \left( P_n(\cos \theta) \right) \frac{1}{k_1r} \{ L_{on} [k_1rj_n(k_1r)]' + M_{on} [k_1rj_n(k_1r)]' \} e^{-j\omega t} \]
\[ = - \sum_{n=0}^{\infty} P_n^1(\cos \theta) \{ L_{on} [k_1rj_n(k_1r)]' + M_{on} [k_1rj_n(k_1r)]' \} e^{-j\omega t} \]  
(84)

\[ H_{\phi}^i = \frac{k_1}{j\omega \mu_1} \sum_{n=0}^{\infty} P_n^1(\cos \theta) \{ L_{on} j_n(k_1r) + M_{on} j_n(k_1r) \} e^{-j\omega t}. \]  
(85)

The field components outside the sphere are
\[ E_{r}^e = - \sum_{n=0}^{\infty} n (n + 1) N_{on} P_n(\cos \theta) \frac{h_n^{(1)}(k_0r)}{k_0r} e^{-j\omega t} \]  
(86)
\[ E_{\theta}^e = - \sum_{n=0}^{\infty} P_n^1(\cos \theta) \frac{1}{k_0} N_{on} [k_0rj_n^{(1)}(k_0r)]' e^{-j\omega t} \]  
(87)
\[ H_{\phi}^e = \frac{k_0}{j\omega \mu_0} \sum_{n=0}^{\infty} P_n^1(\cos \theta) N_{on} h_n^{(1)}(k_0r) e^{-j\omega t}. \]  
(88)

Applying the boundary conditions that
(i) \( E_{\theta}^e = E_{\theta}^i \) and \( H_{\phi}^e = H_{\phi}^i \) at \( r = b \)
and
(ii) \( E_{\theta}^i = 0 \) at \( r = a \),
we obtain
\[ \frac{1}{k_1 b} \{ L_{on} [k_1bj_n(k_1b)]' + M_{on} [k_1bj_n(k_1b)]' \} + \frac{C_{on}(b)}{k_1} \]
\[ = \frac{N_{on}}{k_0 b} [k_0bh_n^{(1)}(k_0b)]' \]  
(89)
\[ k_1 \mu_1 \{ L_{on} j_n(k_1b) + M_{on} j_n(k_1b) \} = k_0 \mu_0 N_{on} h_n^{(1)}(k_0b) \]  
(90)
and
\[ \frac{1}{k_1 a} \{ L_{on} [k_1aj_n(k_1a)]' + M_{on} [k_1aj_n(k_1a)]' \} + \frac{C_{on}(a)}{k_1} = 0. \]  
(91)

Using equation (81) in equations (89), (90) and (91), the amplitude coefficients \( L_{on}, M_{on} \) and \( N_{on} \) can be uniquely determined for each value of \( n \).
The field is thus uniquely determined both inside and outside the dielectric coated conducting sphere for each value of \( n \). This shows that \( M_{0n} \) modes exist for \( n = 0, 1, 2 \), for this structure.

If

\[
Z_n = \begin{bmatrix}
\frac{1}{k_1} [k_1 b j_n (k_1 b)]', & \frac{1}{k_1} [k_1 b y_n (k_1 b)]', & -\frac{1}{k_0} [k_0 b h_n(1) (k_0 b)]' \\
\frac{k_1}{\mu_1} j_n (k_1 b), & \frac{k_1}{\mu_1} y_n (k_1 b), & -\frac{k_0}{\mu_0} h_n(1) (k_0 b) \\
\frac{1}{k_1 a} [k_1 a j_n (k_1 a)]', & \frac{1}{k_1 a} [k_1 a y_n (k_1 a)]', & 0
\end{bmatrix}
\]

then when \( Z_n = 0 \), free oscillations of the dielectric coated conducting sphere results. Hence the roots of the equation \( Z_n = 0 \) determine the characteristic or resonant frequencies of the natural modes of oscillations.

It can similarly be shown that symmetric \( TE_{0n} \) modes also exist for \( n = 0, 1, 2, \ldots \) for the dielectric coated conducting sphere.

6. Conclusions

The following conclusions can be drawn from the above investigations on the dielectric-coated conducting sphere:

(i) It is not possible to excite unsymmetric \( TM \) and \( TE \) modes on the dielectric coated conducting sphere.

(ii) It is possible to excite symmetric \( TM \) and \( TE \) modes, as well as symmetric and unsymmetric hybrid modes on the structure.

Numerical calculations and experimental verification of the results obtained will be reported in subsequent papers.

Acknowledgements

The author wishes to express her gratitude to Dr. S. Dhawan, Director of the Indian Institute of Science, Bangalore, for providing all facilities for carrying out the investigations. She is also grateful to the C.S.I.R. for the sanction of a research scheme on the above subject.

References


10. Chatterjee, R., Keshava Murthy, T. L. and Vedavathy, T. S. Dielectric-coated metal spherical antennas excited in the unsymmetric hybrid mode at microwave frequencies, under publication in the *Journal of Institution of Telecommunication and Electronic Engineers*, India.