THEORY OF OPEN RESONATOR WITH AN AXIAL DIELECTRICAL ROD*

By N. Narasimhan, V. C. Ananthan, and S. K. Chatterjee

(Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore-12 India)

[Received: November 22, 1971]

1. ABSTRACT

The Q factor of a microwave resonator consisting of an uniform circular dielectric rod excited by \( HE_{11} \) and \( E_{01} \) modes and terminated at both ends by circular metal plates has been derived. The constant percentage power contour for both the modes show that most of the energy is located within the resonator and very little is lost by radiation outside the resonator.

2. INTRODUCTION

Theoretical and experimental studies of microwave resonator, closed and open types and resonator consisting of an axial uniform and corrugated metal rod have been made and reported elsewhere, by Chatterjee, et.al. The object of the present paper is to derive an expression for the Q-factor of a resonator consisting of an uniform circular dielectric rod and excited in \( HE_{11} \) and \( E_{01} \) modes and terminated at both ends by circular metal plates of diameter much larger than the diameter of the dielectric rod. All the other sides of the resonator are open. It is also the object to determine the constant percentage power contours round the rod and show that the loss of the energy by radiation is insignificantly small when the diameter of the end-plates are much larger compared to the wavelength of excitation.

3. FIELD COMPONENTS

The field components inside and outside the dielectric rod when it is excited by \( HE_{11} \) mode are as follows (See Fig. 1).

* The project is supported by PL-480, Contract No. E-262-69(N), dated August 30, 1969.
For \( \rho \in \text{Medium 1} \) (Inside the rod):

\[
\begin{align*}
E_{\rho 1} &= -B \left[ \frac{1}{\rho} J_1(k_1 \rho) + \frac{b \beta k_1}{B \omega \epsilon_1 J_1'(k_1 \rho)} \right] \sin \phi e^{i\beta z} \\
E_{\phi 1} &= -B \left[ k_1 J_1'(k_1 \rho) + \frac{b}{B} \frac{\beta}{\omega \epsilon_1} \frac{1}{\rho} J_1(k_1 \rho) \right] \cos \phi e^{i\beta z} \\
E_{z 1} &= -B \left[ \frac{b}{B} \frac{k_1^2}{j \omega \epsilon_1} - J_1(k_1 \rho) \right] \sin \phi e^{i\beta z} \\
H_{\rho 1} &= -B \left[ \frac{\beta k_1}{\omega \mu_0} J_1'(k_1 \rho) + \frac{b}{B} \frac{1}{\rho} J_1(k_1 \rho) \right] \cos \phi e^{i\beta z} \\
H_{\phi 1} &= -B \left[ \frac{1}{\rho} \frac{\beta}{\omega \mu_0} J_1(k_1 \rho) + \frac{b}{B} k_1 J_1'(k_1 \rho) \right] \sin \phi e^{i\beta z}
\end{align*}
\]
For $\rho \geq a$ Medium 2 (outside the rod):

\[
\begin{align*}
E_{x2} &= -C \left[ \frac{1}{\rho} H^{(1)}_1(k_2 \rho) + \frac{c}{C} \frac{\beta k_2}{\omega \epsilon_0} H^{(1)'}_1(k_2 \rho) \right] \sin \phi e^{-j\beta z} \\
E_{\phi 2} &= -C \left[ k_2 H^{(1)}_1(k_2 \rho) + \frac{c}{C} \frac{1}{\rho} \frac{\beta}{\omega \epsilon_0} H^{(1)}_1(k_2 \rho) \right] \cos \phi e^{-j\beta z} \\
E_{z2} &= C \left[ \frac{c}{C} \frac{k_2^2}{j \omega \epsilon_0} H^{(1)}_1(k_2 \rho) \right] \sin \phi e^{-j\beta z} \\
H_{\phi 2} &= C \left[ \frac{\beta k_2}{\omega \mu_0} H^{(1)'}_1(k_2 \rho) + \frac{c}{C} \frac{1}{\rho} H^{(1)}_1(k_2 \rho) \right] \cos \phi e^{-j\beta z} \\
H_{z2} &= -C \left[ (k_2^2/j \omega \mu_0) H^{(1)}_1(k_2 \rho) \right] \cos \phi e^{-j\beta z}
\end{align*}
\]

4. **Boundary Conditions**

In order to derive the characteristic equation and determine the propagation constant, the following boundary conditions are applied at the interface ($\rho = a$) between the dielectric and air.

\[
\begin{align*}
E_{z1} &= E_{z2} ; & H_{z1} &= H_{z2} \\
E_{\phi 1} &= E_{\phi 2} ; & H_{\phi 1} &= H_{\phi 2}
\end{align*}
\]

5. **Characteristic Equation**

By using the proper field components and appropriate boundary conditions, the following characteristic equation is obtained.

\[
\begin{align*}
\left[ \frac{1}{x_1} J'_1(x_1) - \frac{1}{x_2} H^{(1)'}_1(x_2) \right] \left[ \epsilon_{r1} J'_1(x_1) - \frac{1}{x_2} \frac{H^{(1)}(x_2)}{H^{(1)}_1(x_2)} \right] &= \frac{(x_1^2 - x_2^2)(x_1^2 - \epsilon_{r1} x_2^2)}{x_1^4 x_2^4}
\end{align*}
\]
Theory of Open Resonator with an Axial Dielectric Rod

Where, \( x_1 = k_1 a, \quad x_2 = k_2 a \)

\[
\begin{align*}
\varepsilon_{r1} &= \frac{\varepsilon_1}{\varepsilon_0} \\
\left( x_1^2 + (x_2/l)^2 \right)^{1/2} &= \left( \frac{2}{\lambda_0} \right)^2 \quad (\varepsilon_{r1} = 1)
\end{align*}
\]

\[ [5] \]

6. Excitation Constants

The excitation constants \( B, C, b \) and \( c \) are related as follows.

\[
\frac{C}{B} = \frac{x_1^2}{x_2^2} \frac{J_1(x_1)}{H_1^{(1)}(x_1)}
\]

\[
\frac{C}{B} = \frac{x_1^2}{x_2^2} \frac{1}{\varepsilon_{r1}} \frac{J_1(x_1)}{H_1^{(1)}(x_2)}
\]

\[
\frac{b}{B} = \frac{\beta \varepsilon_{r1} x_1^2 - x_2^2}{\omega \mu_0 x_1^2 x_2^2} \left[ \frac{\varepsilon_1 J_1'(x_1)}{x_1 J_1(x_1)} - \frac{\varepsilon_0 H_1''(x_2)}{x_2 H_1^{(1)}(x_2)} \right]^{-1}
\]

\[
\frac{c}{C} = \frac{\beta \varepsilon_0 x_1^2 - x_2^2}{\omega \mu_0 x_1^2 x_2^2} \left[ \frac{\varepsilon_1 J_1'(x_1)}{x_1 J_1(x_1)} - \frac{\varepsilon_0 H_1''(x_2)}{x_2 H_1^{(1)}(x_2)} \right]
\]

\[ [6] \]

The above relations are obtained from the field components and using boundary conditions.

7. Standing Waves

The standing waves formed along the rod due to reflections taking place at both ends of the rod are represented on a vector basis as follows.

\[
\begin{align*}
E_{zs} &= E_{z+} + E_{z-} \quad & H_{zs} &= H_{z+} + H_{z-} \\
E_{ps} &= E_{p+} + E_{p-} \quad & H_{ps} &= H_{p+} + H_{p-} \\
E_{\phi s} &= E_{\phi+} + E_{\phi-} \quad & H_{\phi s} &= H_{\phi+} + H_{\phi-}
\end{align*}
\]

where,

\[
\begin{align*}
E_{z+} &= +E_{z-} \quad & H_{z+} &= -H_{z-} \\
E_{p+} &= -E_{p-} \quad & H_{p+} &= +H_{p-} \\
E_{\phi+} &= -E_{\phi-} \quad & H_{\phi+} &= +H_{\phi-}
\end{align*}
\]

\[ [8] \]
The field components of the standing waves are therefore, for \( \rho = a \).

\[
E_{\rho z} = -2B \left[ \frac{1}{\rho} J_1(k_1 \rho) + \frac{b}{B} \frac{\beta k_1}{\omega \epsilon_1} j' \left( k_1 \rho \right) \right] \sin \phi \sin \beta z \\
E_{\phi z} = -2B \left[ k_1 j' \left( k_1 \rho \right) + \frac{b}{B} \frac{\beta}{\omega \epsilon_1} J_1 \left( k_1 \rho \right) \right] \cos \phi \sin \beta z \\
E_{z1z} = 2B \left[ \frac{b}{B} \frac{k^2_1}{j \omega \epsilon_1} J_1 \left( k_1 \rho \right) \right] \sin \phi \cos \beta z \\
H_{\rho z} = 2B \left[ \frac{k_1 \beta}{\omega \mu_0} j' \left( k_1 \rho \right) + \frac{b}{B} \frac{1}{\rho} J_1 \left( k_1 \rho \right) \right] \cos \phi \cos \beta z \\
H_{\phi z} = -2B \left[ \frac{1}{\rho} \frac{\beta}{\omega \mu_0} J_1 \left( k_1 \rho \right) + \frac{b}{B} \frac{k_1}{\rho} j' \left( k_1 \rho \right) \right] \sin \phi \cos \beta z \\
H_{z1z} = -2B \left[ \frac{k^2_1}{j \omega \mu_0} J_1 \left( k_1 \rho \right) \right] \cos \phi \sin \beta z
\]

[9] The components in the region outside the rod \( \rho = a \) can be written similarly.

8. RESONANT WAVE

The field components \( E_{\rho} \) and \( E_{\phi} \) at \( z=0 \) and \( z=1 \) satisfy the conditions \( E_{\phi} = 0 \) and \( E_{\rho} = 0 \) which lead to \( \sin \beta z = 0 \) which requires \( \beta l = m\pi \) where, \( m \) is a positive integer \( (1, 2, 3 \ldots) \) indicating the number of half cycle variation of the field components in the \( z \)-direction. The components of the resonant waves are obtained substituting \( \beta = (m\pi/l) \) in the equations for standing waves.

For \( \rho \leq a \)

\[
E_{\rho z} = -2B \left[ \frac{1}{\rho} J_1(k_1 \rho) + \frac{b}{B} \frac{\beta k_1}{\omega \epsilon_1} j' \left( k_1 \rho \right) \right] \sin \phi \sin \left( \frac{m\pi}{l} \right) z \\
E_{\phi z} = -2B \left[ k_1 j' \left( k_1 \rho \right) + \frac{b}{B} \frac{1}{\rho} \frac{\beta}{\omega \epsilon_1} J_1 \left( k_1 \rho \right) \right] \cos \phi \sin \left( \frac{m\pi}{l} \right) z \\
E_{z1z} = -2B \left[ \frac{b}{B} \frac{k^2_1}{j \omega \epsilon_1} J_1 \left( k_1 \rho \right) \right] \sin \phi \cos \left( \frac{m\pi}{l} \right) z
\]
Theory of Open Resonator with an Axial Dielectric Rod

\[ H_{r1r} = 2B \left[ \frac{k_1}{j\omega \mu_0} J_1'(k_1 \rho) + \frac{b}{B} \frac{1}{\rho} J_1(k_1 \rho) \right] \cos \phi \cos \left( \frac{m \pi}{l} \right) z \]

\[ H_{\phi 1r} = -2B \left[ \frac{1}{\rho} \frac{\beta}{\mu_0} J_1(k_1 \rho) + \frac{h}{B} k_1 J_1'(k_1 \rho) \right] \sin \phi \cos \left( \frac{m \pi}{l} \right) z \]

\[ H_{z1r} = -2B \left[ \frac{k_1^2}{j\omega \mu_0} J_1(k_1 \rho) \right] \cos \phi \sin \left( \frac{m \pi}{l} \right) z \]

\[ \rho \equiv a; \]

\[ E_{p2r} = -2C \left[ \frac{1}{\rho} \frac{\beta k_2}{\omega \varepsilon_0} H_1^{(2)}(k_2 \rho) + \frac{c}{C} \frac{1}{\omega \varepsilon_0} \frac{1}{\rho} H_1^{(1)}(k_2 \rho) \right] \sin \phi \sin \left( \frac{m \pi}{l} \right) z \]

\[ E_{\phi 2r} = -2C \left[ k_2 H_1^{(1)'}(k_1 \rho) + \frac{c}{C} \frac{\beta k_2}{\omega \varepsilon_0} \frac{1}{\rho} H_1^{(1)}(k_2 \rho) \right] \cos \phi \sin \left( \frac{m \pi}{l} \right) z \]

\[ E_{z2r} = 2C \left[ \frac{c}{C} \frac{k_2^2}{j\omega \mu_0} H_1^{(1)}(k_1 \rho) \right] \sin \phi \cos \left( \frac{m \pi}{l} \right) z \]

\[ H_{\phi 2s} = -2C \left[ \frac{1}{\rho} \frac{\beta}{\omega \mu_0} H_1^{(1)}(k_2 \rho) + \frac{c}{C} \frac{1}{\rho} H_1^{(1)}(k_2 \rho) \right] \cos \phi \cos \left( \frac{m \pi}{l} \right) z \]

\[ H_{z2r} = -2C \left[ \frac{k_2^2}{j\omega \mu_0} H_1^{(1)}(k_2 \rho) \right] \cos \phi \sin \left( \frac{m \pi}{l} \right) z \]

If the field inside the resonator is completely described by equations [10] and [11], the mode of oscillation is pure \( HE_{11} \). If the frequency of excitation or the distance between the end plates is so adjusted that the above conditions are satisfied, the cavity formed by the structure with the two end plates is said to be in resonance

9. Magnetic Energy Stored

The energy stored in the magnetic field consists of two parts \( W_{M1} \), in the field inside the rod and \( W_{M2} \) in the field outside the rod (\( W_{M2} \)). The total energy \( W_M \) stored in the magnetic field is therefore,

\[ W_M = W_{M1} + W_{M2} \]

\[ = \frac{\mu_0}{2} \int_0^a \int_0^{2\pi} \int_0^l |H_1|^2 \rho \, d\rho \, d\phi \, dz \]

\[ + \frac{\mu_0}{2} \int_a^d \int_0^{2\pi} \int_0^l |H_2|^2 \rho \, d\rho \, d\phi \, dz \]
where,
\[ |H_1|^2 = |H_{\rho 1r}|^2 + |H_{\phi 1r}|^2 + |H_{z 1r}|^2 \]
\[ |H_2|^2 = |H_{\rho 2r}|^2 + |H_{\phi 2r}|^2 + |H_{z 2r}|^2 \]  

[13]

The energy stored in the two media are:

\[ WM_1 = \frac{\mu_0}{2} \pi \frac{1}{2} 4B^2 \left\{ \int_{\rho = 0}^{a} \left( \frac{k_1 B}{\omega \mu_c} \right)^2 \rho [J_1(k_1 \rho)]^2 \, d\rho \right. \]
\[ + \int_{\rho = 0}^{a} \left( \frac{b}{B} \right)^2 \frac{1}{\rho} [J_1(k_1 \rho)]^2 \, d\rho + \int_{\rho = 0}^{a} \frac{2b \beta k_1}{B \omega \mu_0} J_1(k_1 \rho) J_1'(k_1 \rho) \, d\rho \]
\[ + \int_{\rho = 0}^{a} \left( \frac{\beta}{\omega \mu_0} \right)^2 \frac{1}{\rho} [J_1(k_1 \rho)]^2 \, d\rho + \int_{\rho = 0}^{a} \left( \frac{b}{B} \right)^2 k_1^2 \rho [J_1'(k_1 \rho)]^2 \, d\rho \]
\[ + \int_{\rho = 0}^{a} \frac{2b \beta k_1}{B \omega \mu_0} J_1'(k_1 \rho) J_1(k_1 \rho) \, d\rho - \left( \frac{k_1^2}{\omega \mu_0} \right)^2 \int_{\rho = 0}^{a} \rho [J_1(k_1 \rho)]^2 \, d\rho \]

which results in

\[ WM_1 = \pi \mu_0 B I^2 \left[ \left\{ \frac{\beta}{\omega \mu_0} \right\}^2 + \left( \frac{b}{B} \right)^2 \left( \frac{k_1}{\omega \mu_0} \right)^2 \right] \left\{ \frac{(k_1 a)^2}{\omega \mu_0} \right\} \left\{ [J_0(k_1 a)]^2 \right. \]
\[ + [J_1(k_1 a)]^2 \right\} - \left( \frac{\beta}{\omega \mu_0} - \frac{b}{B} \right)^2 \left\{ J_1(k_1 a) \right\}
\[ \left. + \left( \frac{k_1}{\omega \mu_0} \right)^2 \left\{ k_1 a J_0(k_1 a) J_1(k_1 a) \right\} \right\} \]  

[14]

Similarly,

\[ WM_2 = \pi \mu_0 C I^2 \left[ \left\{ \frac{\beta}{\omega \mu_0} \right\}^2 + \left( \frac{c}{C} \right)^2 \left( \frac{k_2}{\omega \mu_0} \right)^2 \right] \left\{ \frac{(k_2 d)^2}{\omega \mu_0} \right\} \left\{ [H_0^{(1)}(k_2 d)]^2 \right. \]
\[ + [H_1^{(1)}(k_2 d)]^2 \right\} - \frac{(k_2 a)^2}{2} \left\{ [H_0^{(1)}(k_2 a)]^2 - [H_1^{(1)}(k_2 a)]^2 \right\} \]
\[ - \left( \frac{\beta}{\omega \mu_0} - \frac{c}{C} \right)^2 \left\{ [H_1^{(1)}(k_2 d)]^2 - [H_1^{(1)}(k_2 a)]^2 \right\}
\[ + \left( \frac{k_2}{\omega \mu_0} \right)^2 \left\{ (k_2 d) H_0^{(1)}(k_2 d) H_1^{(1)}(k_2 d) - (k_2 a) H_0^{(1)}(k_2 a) H_1^{(1)}(k_2 a) \right\} \]  

[15]
10. POWER LOSS

The total power lost $P$ is composed of the power lost $P_d$ in the dielectric rod, power lost in the end plates $P_e$ and power lost by radiation $P_R$ outside the resonator, i.e.,

$$P = P_d + P_e + P_R$$  \[16\]

If the diameters $d$ of the end plates is much larger than the diameter $(2a)$ of the rod, the loss by radiation $P_R$ outside the resonator can be ignored compared to the other two terms.

The power lost in the dielectric rod is given by

$$P = \frac{1}{2} \int_{\phi=0}^{\pi} \int_{\rho=0}^{a} \sigma_1 |E|^2 \rho \, d\rho \, d\phi$$  \[17\]

where $\sigma_1 = \omega \epsilon \tan \delta$ and

$$|E|^2 = |E_{\rho1}|^2 + |E_{\phi1}|^2 + |E_{z1}|^2$$

Substituting the values of the field components, equation \[17\] becomes

$$P_d = \sigma_1 \frac{1}{2} \pi B^2 \left[ \int_{\rho=0}^{a} \frac{1}{\rho} |J_1(k_1 \rho)|^2 \, d\rho + \int_{\rho=0}^{a} \left( \frac{b \beta k_1}{B \omega \epsilon} \right)^2 |J_1'(k_1 \rho)|^2 \, d\rho \right.$$  

$$+ \int_{\rho=0}^{a} \left( \frac{b \beta k_1}{B \omega \epsilon} \right)^2 |J_1'(k_1 \rho)|^2 \, d\rho + \int_{\rho=0}^{a} \left( \frac{b \beta k_1}{B \omega \epsilon} \right)^2 |J_1'(k_1 \rho)|^2 \, d\rho$$

$$+ \int_{\rho=0}^{a} \left( \frac{b k_1^2}{B \omega \epsilon} \right)^2 \rho |J_1(k_1 \rho)|^2 \, d\rho \right]$$

$$- \frac{\pi l}{2} \sigma_1 B^2 \left[ \left( \frac{(k_1 a)}{2} \right)^2 \left( 1 + \left( \frac{b \beta}{B \omega \epsilon} \right)^2 \right) - \frac{1}{2} \left( \frac{b k_1^2 a}{B \omega \epsilon} \right)^2 \right] \{J_0(k_1 a)\}^2$$

$$+ \{J_1(k_1 a)\}^2 \left[ 1 - \left( \frac{b \beta}{B \omega \epsilon} \right)^2 \right] + \left( \frac{k_1 a}{2} \right)^2 \left( 1 + \left( \frac{b \beta}{B \omega \epsilon} \right)^2 \right) - \frac{1}{2} \left( \frac{b k_1^2 a}{B \omega \epsilon} \right)^2 \right]$$

$$+ \{J_0(k_1 a) J_1(k_1 a)\} \left\{ \frac{1}{k_1 a} \left( \frac{b k_1^2 a}{B \omega \epsilon} \right) \right\}$$  \[18\]
The power lost in the two end plates is

\[ P = 2 \times \frac{1}{2 \sqrt{2}} \left( \frac{\omega \mu_0}{\sigma_e} \right)^{1/2} \int \int_{\rho=a}^{d} \left| H_{\text{tan}} \right|^2 \rho d\rho \cdot d\phi \]  

where,

\[ \sigma_e = \text{conductivity of the end plates} \]

\[ |H_{\text{tan}}|^2 = |H_{\rho 2}|^2 + |H_{\phi 2}|^2 \]

Substituting the value of the field components, equation [19] after integration yields.

\[ P = \pi C^2 \left( \frac{\omega \mu_0}{2 \sigma_0} \right)^{1/2} \left[ \left( \frac{\beta k_2}{\omega \mu_0} \right)^2 + \left( \frac{ck_2}{C} \right)^2 \right] \left\{ \frac{d^2}{2} \left[ H_0^{(1)}(k_2 d) \right]^2 - \frac{a^2}{2} \left[ H_0^{(1)}(k_2 a) \right]^2 \right\} \]

\[ + \left[ \left( \frac{\beta k_2}{\omega \mu_0} \right)^2 + \left( \frac{ck_2}{C} \right)^2 + \left( \frac{\beta}{\omega \mu_0} - \frac{c}{C} \right)^2 \right] \left\{ \frac{d^2}{2} \left[ H_1^{(1)}(k_2 d) \right]^2 - \frac{a^2}{2} \left[ H_1^{(1)}(k_2 a) \right]^2 \right\} \]

\[ \text{[20]} \]

11. 'Q' Factor

The quality factor \( Q \) of the resonator is defined as

\[ Q = \omega \cdot \frac{W_{M1} + W_{M2}}{P_{\text{total}}} = \omega \cdot \frac{W_{M1} + W_{M2}}{P_e + P_d} \]  

Substituting appropriate expressions for the maximum energy stored in the magnetic field and the total power lost, the expression for \( Q \) becomes

\[ Q = \frac{2 \pi f (\mu_0/\varepsilon_0) [X + (C^2/B^2)Y]}{\left( \frac{1}{2} \omega \varepsilon_r \tan \delta \right) (U + V) + (C^2/B^2) (1/\varepsilon_0) (\omega \mu_0/2 \sigma_e)^{1/2} W} \]  

\[ \text{[22]} \]

where

\[ X = \left\{ \left( \frac{\beta}{\omega \mu_0} \right)^2 \right\} + \left\{ \left( \frac{b}{B} \right)^2 \right\} - \left\{ \left( \frac{k_1 a}{\omega \mu_0} \right)^2 \right\} \left\{ \left( \frac{k_1 a}{2} \right)^2 \right\} \left\{ \{ J_0(k_1 a) \}^2 + \{ J_1(k_1 a) \}^2 \right\} \]

\[ - \left\{ \left( \frac{\beta}{\omega \mu_0} - \frac{b}{B} \right)^2 \right\} \left\{ \{ J_1(k_1 a) \}^2 \right\} + \left\{ \left( \frac{k_1 a}{\omega \mu_0} \right)^2 \right\} \left\{ (k_1 a) J_0(k_1 a) J_1(k_1 a) \right\} \]
The values of radial propagation constants \( k_1 \) and \( k_2 \) are found from the values of \( x_1 \) and \( x_2 \) which are determined by solving the characteristic equation (4) with the aid of equation (5). The variation of radial propagation constants with \((2a/\lambda_0)\) is shown in Fig. 2. Fig. 3 shows the variation of axial phase constant with \((2a/\lambda_0)\) which is determined from the relation between \( k \) and \( \beta \).

13. Evaluation of \( Q \)

The values of \( Q \) of the resonator as a function of \((2a/\lambda)\), \( L \), \( \tan \delta \), \( \sigma_e \) and \( \lambda \) calculated from equation (22) with proper values of \( k_1 \), \( k_2 \) and \( \beta \) are plotted in figures 4 - 8 respectively.
14. FIELD COMPONENTS $E_{01}$ - MODE

The field components inside ($\rho \leq a$) and outside the rod ($\rho \geq a$) are respectively

Medium 1: $\rho = a$

$$E_{\rho 1} = \frac{Bk_1}{\omega \epsilon_1} J_1 (k_1 \rho) \exp (-j \beta z)$$

$$E_{z1} = \frac{Bk_1^2}{j \omega \epsilon_1} J_0 (k_1 \rho) \exp (-j \beta z)$$

$$H_{\phi 1} = Bk_1 J_1 (k_1 \rho) \exp (-j \beta z)$$

[24]
Fig. 4

Fig. 5

L = 50.0 cm
λ = 3.14 cm
ε_r = 2.56

a = 0.4710 cm
λ = 3.14 cm
ε_r = 2.56
Theory of Open Resonator with an Axial Dielectric Rod

Fig. 6

\[ a = 0.5 \text{ cm} \]
\[ \lambda = 3.14 \text{ cm} \]
\[ \varepsilon_r = 2.56 \]

Fig. 7

\[ a = 0.5 \text{ cm} \]
\[ \lambda = 3.14 \text{ cm} \]
\[ \varepsilon_r = 2.56 \]

Fig. 8

\[ a = 0.5 \text{ cm} \]
\[ L = 50.0 \text{ cm} \]
\[ \varepsilon_r = 2.56 \]
Medium 2: \( \rho = a \)

\[ E_{z2} = \frac{C k_2^2 \beta}{\omega \epsilon_0} H_1^{(1)}(k_2 \rho) \exp(-j \beta z) \]

\[ E_{z2} = \frac{C k_2^2}{j \omega \epsilon_0} H_1^{(1)}(k_2 \rho) \exp(-j \beta z) \]

\[ H_{\phi_2} = C k_2 H_1^{(1)}(k_2 \rho) \exp(-j \beta z) \]

15. **Standing Waves**

In this case, the standing waves are represented by

\[ E_{z+} = E_{z-} + E_{z-}, \quad E_{\rho+} = E_{\rho-} + E_{\rho-}, \quad H_{\phi+} = H_{\phi-} + H_{\phi-} \]

where,

\[ E_{z+} = E_{z-} \quad E_{\rho+} = E_{\rho-} \quad H_{\phi+} = H_{\phi-} \]

16. **Resonant Waves**

Due to the vanishing of the tangential components of the electric field at the two end plates \( z = 0 \) and \( z = l \), \( \beta = n \pi/1 \), where \( n \) is a positive integer. The field components of the resonant waves are

Medium 1: \( \rho \leq a \)

\[ E_{\rho 1r} = 2 j B \frac{k_1 \beta}{\omega \epsilon_1} J_1(k_1 \rho) \sin \left( \frac{n \pi z}{l} \right) \]

\[ E_{z1r} = B \frac{k_1^2}{j \omega \epsilon_1} J_0(k_1 \rho) \cos \left( \frac{n \pi z}{l} \right) \]

\[ H_{\phi 1r} = 2 B k_1 J_1(k_1 \rho) \cos \left( \frac{n \pi z}{l} \right) \]

Medium 2: \( \rho = a \)

\[ E_{\rho 2r} = 2 C \frac{k_2 \beta}{\omega \epsilon_0} H_1^{(1)}(k_2 \rho) \sin \left( \frac{n \pi z}{l} \right) \]

\[ E_{z2r} = 2 C \frac{k_2^2}{j \omega \epsilon_0} H_0^{(1)}(k_2 \rho) \sin \left( \frac{n \pi z}{l} \right) \]

\[ H_{\phi 2r} = 2 C k_2 H_1^{(1)}(k_2 \rho) \cos \left( \frac{n \pi z}{l} \right) \]
17. Maximum Energy Stored in Magnetic Field

The total maximum energy stored in the magnetic field is

\[ W_M = W_{M1} + W_{M2} \]

where, the magnetic energy stored in medium 1 \( (W_{M1}) \) and in medium 2 \( (W_{M2}) \) are respectively

\[ W_{M1} = \frac{\mu_0}{2} \int \left| H_{\text{max}} \right|^2 dv \]

\[ W_{M1} = \frac{\mu_0}{2} \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \int_{\zeta=0}^{1} 4B^2 k_1^2 \left[ J_1(k_1 \rho) \right]^2 \cos^2 \left( \frac{n \pi z}{l} \right) \rho \, d\rho \, d\phi \, dz \]

\[ = \pi \mu_0 B^2 k_1^2 l a^2 \left[ J_0(k_1 a)^2 + J_1(k_1 a)^2 - \frac{2}{k_1 a} J_0(k_1 a) J_1(k_1 a) \right] \quad [30] \]

\[ W_{M2} = \pi \mu_0 C^2 k_2^2 l \left[ \frac{d^2}{a^2} \left[ H_0^{(1)}(k_2 d)^2 + H_1^{(1)}(k_2 d)^2 \right] - \frac{2}{k_2 a} H_0^{(1)}(k_2 a) H_1^{(1)}(k_2 a) \right] \]

\[ - a^2 \left[ H_0^{(1)}(k_2 a)^2 + H_1^{(1)}(k_2 a)^2 - \frac{2}{k_2 a} H_0^{(1)}(k_2 a) H_1^{(1)}(k_2 a) \right] \quad [31] \]

Therefore, the total maximum energy stored in the magnetic field is

\[ W_M = \pi \mu_0 l a^2 B^2 \left[ k_1^2 \left\{ J_0(k_1 a)^2 + J_1(k_1 a)^2 - \frac{2}{k_1 a} J_0(k_1 a) J_1(k_1 a) \right\} \right] \]

\[ - \frac{C^2}{B^2} k_2^2 \left( \frac{d^2}{a^2} \left[ H_0^{(1)}(k_2 d)^2 + H_1^{(1)}(k_2 d)^2 \right] - \frac{2}{k_1 a} H_0^{(1)}(k_2 d) H_1^{(1)}(k_2 d) \right] \]

\[ - \left[ H_0^{(1)}(k_2 a)^2 + H_1^{(1)}(k_2 a)^2 - \frac{2}{k_2 a} H_0^{(1)}(k_2 a) H_1^{(1)}(k_2 a) \right] \quad [32] \]

18. Power Loss in the Resonator F01-Mode

The total power loss in the resonator is

\[ P = P_d + P_r \]

[33]
neglecting the loss by radiation outside the resonator. Where, the power loss in the two end plates is

\[ P_e = 2 \times \frac{1}{2\sqrt{2}} \left( \frac{\omega \mu \varepsilon}{\sigma \varepsilon} \right)^{1/2} \int_{\rho=a}^{a} \int_{\phi=0}^{2\pi} |H_{\tan}|^2 \rho \, d\rho \, d\phi \]  \[ \text{[34]} \]

where \(|H_{\tan}| = |H_{\phi 2}|\), \(\sigma\) and \(\mu\) are the conductivity and permeability respectively of the end plates and \(d\) is the radius of the end plate.

Substituting \(H_{\phi 2}\) in the integrand, equation [34] becomes

\[ P_e = \pi k_2^2 C^2 \left( \frac{\omega \mu \varepsilon}{2\sigma \varepsilon} \right)^{1/2} \left[ d^2 \left( |H_{0}^{(1)}(k_2 d)|^2 + |H_{1}^{(1)}(k_2 d)|^2 \right) - \frac{2}{k_2 d} \left( H_{0}^{(1)}(k_2 a) H_{1}^{(1)}(k_2 d) \right) \right. \]
\[ \left. - a^2 \left( |H_{0}^{(1)}(k_2 a)|^2 + |H_{1}^{(1)}(k_2 a)|^2 - \frac{2}{k_2 a} H_{0}^{(1)}(k_2 a) H_{1}^{(1)}(k_2 a) \right) \right] \]  \[ \text{[35]} \]

The power loss in the dielectric rod is

\[ P_d = \frac{1}{2} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \sigma_1 |E|^2 \rho \, d\rho \, d\phi \]  \[ \text{[36]} \]

where, \(\sigma_1 = \omega \varepsilon_1 \tan \delta\) and \(|E|^2 = |E_{\rho 1}|^2 + |E_{\phi 1}|^2\).

Substituting \(E_{\tan}\) in the integrand, equation [36] reduces to

\[ P_d = \frac{\pi}{2} \frac{B^2 k_2^2 a^2 \sigma_1}{\omega^2 \varepsilon_1^2} \left[ \left( \left| J_0 (k_1 a) \right|^2 + \left| J_1 (k_1 a) \right|^2 \right) \left( \beta^2 - k_2^2 \right) \right. \]
\[ \left. - 2 a^2 \frac{\beta^2}{k_1 a} J_0 (k_1 a) J_1 (k_1 a) \right] \]  \[ \text{[36a]} \]

19. \('Q'\) Factor: \(E_{\phi 1}\) - Mode

\[ Q = \omega \left( W_M/P \right) \] gives

\[ Q = \frac{\left( \mu_0/\varepsilon_0 \right) \left[ R_1 + (C/B)^2 R_2 \right] }{\left( \omega \varepsilon_1 \tan \delta \right)/2 |R_3| + (C/B)^2 (1/\varepsilon_0) (\omega \mu_0/2\sigma_\varepsilon)^{1/2} |R_4|} \]  \[ \text{[37]} \]

where, \(R_1\), \(R_2\), \(R_3\) and \(R_4\) are given by the following expressions.

\[ R_1 = a^2 k_2^2 \left( \left| J_0 (k_1 a) \right|^2 + \left| J_1 (k_1 a) \right|^2 - 2 \frac{2}{k_1 a} J_0 (k_1 a) J_1 (k_1 a) \right) \]
Theory of Open Resonator with an Axial Dielectric Rod

\[ R_2 = k^2 \int \left[ \left( \frac{d^2}{d} \left( H_0^{(1)}(k_2 d) - H_1^{(1)}(k_2 d) \right) \right)^2 + \left( \frac{d^2}{d} \left( H_1^{(1)}(k_2 d) \right) \right)^2 - \frac{2}{k_2 d} \frac{H_0^{(1)}(k_2 d)}{H_1^{(1)}(k_2 d)} \right] \]

\[ R_3 = \left( \frac{k_1}{\omega \varepsilon_r} \right)^2 \left\{ \left[ J_0(k_1 a) \right]^2 + \left[ J_1(k_1 a) \right]^2 \right\} + \left\{ \left( \frac{\beta^2 - k_1^2}{\beta^2} \right) - \frac{2}{k_1} \frac{J_0(k_1 a)}{J_1(k_1 a)} \right\} \]

\[ R_4 = k^2 \int \left[ \left( \frac{d^2}{d} \left( H_0^{(1)}(k_2 d) - H_1^{(1)}(k_2 d) \right) \right)^2 + \left( \frac{d^2}{d} \left( H_1^{(1)}(k_2 d) \right) \right)^2 - \frac{2}{k_2 d} \frac{H_0^{(1)}(k_2 d)}{H_1^{(1)}(k_2 d)} \right] \]

The value of \( C/B \) is given by the following expression

\[ (C/B)^2 = \left[ \frac{k^2}{k_1} \frac{1}{\varepsilon_r} \frac{J_0(k_1 a)}{H_1^{(1)}(k_2 d)} \right]^2 \]

The variation of \( Q \) for \( E_0 \) mode as a function of \( 2a/\lambda_0 \), \( L \), \( \tan \delta \) and \( \sigma \), are shown in Fig. 9—13.

20. Constant Percentage Power Contours

In calculating the \( Q \)-factor of the resonator, radiation loss outside the resonator has been assumed to be negligible. This is justified if most of the energy is located inside the resonator, i.e., within a radius \( d \) from the axis of the resonator where \( d \) is the radius of the end plates. This can be determined by calculating the constant percentage power contour.

The amount of relative power flow outside the guide can be represented from the constant percentage power contours round the guide which are determined as follows.

If \( \rho = r_1, r_2, r_3 \cdots r_n \) represent the radii of the circles representing the contours inside which constant powers \( P_{z1}, P_{z2}, P_{z3} \cdots P_{zn} \) flowing along the rod are located, then the ratio of powers with respect to the total power \( P_{zn} \) is

\[ P_{zi}/P_{zn} = W_i \text{ at } \rho = r_i \]

\[ P_{z2}/P_{zn} = W_2 \text{ at } \rho = r_2 \]

\[ P_{z3}/P_{zn} = W_3 \text{ at } \rho = r_3 \]

\[ \vdots \]

\[ P_{zn}/P_{zn} = W_n \text{ at } \rho = r_n \]
FIG. 9

$L = 50.0 \text{ cm}$

$\lambda = 3.14 \text{ cm}$

$\varepsilon_r = 2.56$

FIG. 10

$\sigma = 1.27 \text{ cm}$

$\lambda = 3.14 \text{ cm}$

$\varepsilon_r = 2.56 \text{ cm}$
Theory of Open Resonator with an Axial Dielectric Rod

\begin{align*}
\alpha & = 1.27 \text{ cm} \\
L & = 50.0 \text{ cm} \\
\varepsilon_r & = 2.56 \\
\lambda & = 3.14 \text{ cm}
\end{align*}

\textbf{FIG. 11}

\begin{align*}
\alpha & = 1.27 \text{ cm} \\
L & = 50.0 \text{ cm} \\
\lambda & = 3.14 \text{ cm} \\
\varepsilon_r & = 2.56
\end{align*}

\textbf{FIG. 12}

\begin{align*}
\alpha & = 1.27 \text{ cm} \\
L & = 50.0 \text{ cm} \\
\lambda & = 3.14 \text{ cm} \\
\varepsilon_r & = 2.56
\end{align*}

\textbf{FIG. 13}
where,
\[ P_{s1} = P_{s1}^l + P_{s1}^0 \] contained within a radius \( \rho = r_1 \).
\[ P_{s2} = P_{s2}^l + P_{s2}^0 \] contained within a radius \( \rho = r_2 \).
\[ P_{s3} = P_{s3}^l + P_{s3}^0 \] contained within a radius \( \rho = r_3 \).
\[ \vdots \]
\[ P_{sn} = P_{sn}^l + P_{sn}^0 \] contained within a radius \( \rho = r_n \).

\( P_{sn} \) represents the total power contained within a contour of radius \( r_n \) and \( r_1 < r_2 < r_3 \cdots < r_n \). The values of \( P_{s1}, P_{s2}, P_{s3}, \ldots P_{sn} \), are determined from equation [46] for \( \text{HE} \) mode and for the \( E_0 \)-mode by replacing the integrals

\[
\int_{p=a}^{p=r_1} \int_{\phi=a}^{\phi=r_1} \cdots \int_{\phi=a}^{\phi=r_n}
\]

respectively.

20.1 Power Contour for \( \text{HE} \) Mode

The power flow in the radial, azimuthal and longitudinal directions are

\[ P_r = \frac{1}{2} \Re \iint_{S} [E_z H^*_r - E_r H^*_z] \rho d\phi d\zeta \] \[39a\]

\[ P_\phi = \frac{1}{2} \Re \iint_{S} [E_z H^*_\phi - E_\phi H^*_z] d\rho d\zeta \] \[39b\]

\[ P_z = \frac{1}{2} \Re \iint_{S} [E_\rho H^*_\phi - E_\phi H^*_\rho] \rho d\rho d\phi \] \[39c\]

The power launched in the dielectric rod will be transmitted in the longitudinal direction, where there is no radiation. But if some power is lost by radiation, the power will not only be transmitted in the longitudinal direction but also in the radial and azimuthal directions. In order to determine how much power is lost by radiation, the power flow \( P_r \) and \( P_\phi \) in the \( \rho \) and \( \phi \) directions respectively are also calculated.
The power flow inside $p^i$ and $P^0$ in the three directions are

$$p^i = B B \left[ \frac{b}{B} \pi k_1 \left( 1 - \frac{\gamma^2}{\omega^2 \mu_0 \epsilon_1} \right) \int_{\rho=0}^{a} J_0(k_1 \rho) J_1(k_1 \rho) \, d\rho \right. \nonumber $$

$$- \frac{\pi \gamma_1 k_1}{j \omega} \left( \frac{1}{\mu_0} + \frac{b^2}{B^2} \frac{1}{\epsilon_1} \right) \int_{\rho=0}^{a} J_0(k_1 \rho) J_1(k_1 \rho) \, d\rho \nonumber $$

$$+ \frac{\pi \gamma_1 k_1}{j \omega} \left( \frac{1}{\mu_0} + \frac{b^2}{B^2} \frac{1}{\epsilon_1} \right) \int_{\rho=0}^{a} \left\{ J_1(k_1 \rho) \right\}^2 \, d\rho \nonumber $$

$$- \frac{b}{B} \pi \left( 1 - \frac{\gamma}{\omega^2 \mu_0 \epsilon_1} \right) \int_{\rho=0}^{a} \left\{ J_1(k_1 \rho) \right\}^2 \, d\rho \nonumber $$

$$+ \frac{\pi \gamma_1 k_1^2}{2 j \omega} \left( \frac{1}{\mu_0} + \frac{b^2}{B^2} \frac{1}{\epsilon_1} \right) \int_{\rho=0}^{a} \rho \left\{ J_0(k_1 \rho) \right\}^2 \, d\rho \right] \tag{40} \nonumber $$

$$P^0 = C C \left[ \frac{c}{C} \pi k_2 \left( 1 - \frac{\gamma^2}{\omega^2 \mu_0 \epsilon_0} \right) \int_{\rho=a}^{\infty} H_0^{(1)}(k_2 \rho) H_1^{(1)}(k_2 \rho) \, d\rho \nonumber $$

$$- \frac{\pi \gamma_2 k_2}{j \omega} \left( \frac{1}{\mu_0} + \frac{c^2}{C^2} \frac{1}{\epsilon_0} \right) \int_{\rho=a}^{\infty} H_0^{(1)}(k_2 \rho) H_1^{(1)}(k_2 \rho) \, d\rho \nonumber $$

$$+ \frac{\pi \gamma_2 k_2}{j \omega} \left( \frac{1}{\mu_0} + \frac{c^2}{C^2} \frac{1}{\epsilon_0} \right) \int_{\rho=a}^{\infty} \frac{1}{\rho} \left[ H_1^{(1)}(k_2 \rho) \right]^2 \, d\rho \nonumber $$

$$- \frac{c}{C} \left( 1 - \frac{\gamma^2}{\omega^2 \mu_0 \epsilon_0} \right) \int_{\rho=a}^{\infty} \frac{1}{\rho} \left\{ H_0^{(1)}(k_2 \rho) \right\}^2 \, d\rho \nonumber $$

$$+ \frac{\pi \gamma_2 k_2^2}{2 j \omega} \left( \frac{1}{\mu_0} + \frac{c^2}{C^2} \frac{1}{\epsilon_0} \right) \int_{\rho=a}^{\infty} \rho \left\{ H_0^{(1)}(k_2 \rho) \right\}^2 \, d\rho \right] \tag{41} \nonumber $$
In calculating $p'_\phi$ and $P^0_\phi$, the limit of $z$ has been taken from $z=0$ to $z=1$ where $I'$ is the length of the dielectric rod $p'_\phi$ and $P^0_\phi$ has been evaluated for $\phi=\pi/4$ as $P^0_\phi=0$ for $\phi=0$, $\pi/2$, $\pi$, $3\pi/2$, and $2\pi$.

$$p'_\phi =\frac{jBB^*}{4} \left[ \frac{b}{B} \frac{2l\beta_1 k_1^2}{\omega^2 \mu_0 \epsilon_1} \int_{\rho=0}^{a} \frac{1}{\rho} \left\{ J_1(k_1 \rho) \right\}^2 d\rho \right. \left. - \frac{b}{B} \frac{2l\beta_1 k_1^2}{\omega^2 \mu_0 \epsilon_1} \int_{\rho=0}^{a} J_0(k_1 \rho) J_1(k_1 \rho) d\rho \right. $${}\left. - \frac{1}{\omega} \left( \frac{1}{\mu_0} + \frac{b^2}{B^2} \right) \int_{\rho=0}^{a} \frac{1}{\rho} \left\{ J_1(k_1 \rho) \right\}^2 d\rho \right]$$ [42]

$$P^0_\phi =\frac{jCC^*}{4} \left[ \frac{c}{C} \frac{2l\beta_2 k_2^2}{\omega^2 \mu_0 \epsilon_0} \int_{\rho=0}^{a} \frac{1}{\rho} \left\{ H_1^{(1)}(k_2 \rho) \right\}^2 d\rho \right. \left. - \frac{c}{C} \frac{2l\beta_2 k_2^2}{\omega^2 \mu_0 \epsilon_0} \int_{\rho=0}^{a} H_1^{(1)}(k_2 \rho) H_1^{(1)}(k_2 \rho) d\rho \right. $${}\left. - \frac{1}{\omega} \left( \frac{1}{\mu_0} + \frac{c^2}{C^2} \right) \int_{\rho=0}^{a} \frac{1}{\rho} \left\{ H_1^{(1)}(k_2 \rho) \right\}^2 d\rho \right]$$ [43]

In calculating $p'_\phi$ and $P^0_\phi$ the limit of $\phi$ has been taken from $\phi=0$ to $\phi=2\pi$ and the limit of $z$ is from $z=0$ to $z=1$. $p'$ and $P^0$ have been evaluated for
The axial propagation constant \( \gamma = j \beta \), and \((\beta_1 = \beta_2 = \beta)\) has been obtained from the solution of the characteristic equation which gives \( k_1 \) and \( k_2 \) and from equation relating \( \beta \) to the radial constants \( k \).

The power flow \( P_i \) and \( P_o \) for different values of \( 2a/\lambda \) have been calculated for perspex rod \((\varepsilon_r = 2.56)\) and the results are tabulated below.

### TABLE 1

**Power flow inside and outside the rod in the \( z \)-direction**

<table>
<thead>
<tr>
<th>( 2a/\lambda )</th>
<th>( P_i ) (Watts/Sq. cm.)</th>
<th>( P_o ) (Watts/Sq. cm.)</th>
<th>( P_z = P_i + P_o ) (Watts/Sq. cm.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.01363 (</td>
<td>B</td>
<td>^2 )</td>
</tr>
<tr>
<td>0.6</td>
<td>0.01349 (</td>
<td>B</td>
<td>^2 )</td>
</tr>
<tr>
<td>0.4</td>
<td>0.029965 (</td>
<td>B</td>
<td>^2 )</td>
</tr>
<tr>
<td>0.3</td>
<td>0.027806 (</td>
<td>B</td>
<td>^2 )</td>
</tr>
</tbody>
</table>

### TABLE 2

**Power flow inside and outside the rod in the radial \( \rho \)-direction**

<table>
<thead>
<tr>
<th>( 2a/\lambda )</th>
<th>( P_i ) Watts/Sq. cm.</th>
<th>( P_o ) Watts/Sq. cm.</th>
<th>( P_z = P_i + P_o ) Watts per sq. cm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>( j \ 0.0^49086 \ 1 \ 1</td>
<td>B</td>
<td>^2 )</td>
</tr>
<tr>
<td>0.6</td>
<td>( -j \ 0.0^6965 \ 1 \ 1</td>
<td>B</td>
<td>^2 )</td>
</tr>
<tr>
<td>0.4</td>
<td>( -j \ 0.0^2031 \ 1 \ 1</td>
<td>B</td>
<td>^2 )</td>
</tr>
<tr>
<td>0.3</td>
<td>( -j \ 0.0^3466 \ 1 \ 1</td>
<td>B</td>
<td>^2 )</td>
</tr>
</tbody>
</table>

### TABLE 3

**Power flow inside and outside the guide in the azimuthal direction**

<table>
<thead>
<tr>
<th>( 2a/\lambda )</th>
<th>( P_i^{\phi} ) Watts/cm²</th>
<th>( P_o^{\phi} ) Watts/cm²</th>
<th>( P_{\phi} = P_i^{\phi} + P_o^{\phi} ) Watts/cm²</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>( -j \ 0.0^46200 \ 1 \ 1</td>
<td>B</td>
<td>^2 )</td>
</tr>
<tr>
<td>0.6</td>
<td>( -j \ 0.0^27099 \ 1 \ 1</td>
<td>B</td>
<td>^2 )</td>
</tr>
<tr>
<td>0.4</td>
<td>( -j \ 0.0^28581 \ 1 \ 1</td>
<td>B</td>
<td>^2 )</td>
</tr>
<tr>
<td>0.3</td>
<td>( -j \ 0.0^6279 \ 1 \ 1</td>
<td>B</td>
<td>^2 )</td>
</tr>
</tbody>
</table>
The power flow $P_s$ and $P_d$ are reactive and hence the relative power flowing inside the guide is expressed as a percentage as follows

$$W = \frac{P_s^1}{P_s^1 + P_d^0} \times 100$$  [46]

The results of computation of $p_s^1/P_s$ as a function of $2a/\lambda_0$ for $\varepsilon_r = 2.56$ and as a function of $2a/\lambda_0 = 0.8$ at $\lambda = 3.2$ cm. for the $H E_{11}$ mode are shown in Figures 14 and 15, respectively. Figure 16 represents graphically the relation between $W\%$ and $P = r_1, r_2, \ldots, r_n$ for different values of $(2a/\lambda_0)$ and $\varepsilon_r = 2.56$. From Fig. 17, the radii of the constant percentage power contours for different values of $(2a/\lambda_0)$ are plotted.

### 20.2 Power contour for $E_0$ mode

The power flow inside and outside the dielectric rod excited in $E_0$ mode has been calculated following the above method,

$$P_s^1 = \frac{\pi B^2 \omega_0^2 k_0^2}{2 \omega \varepsilon_0 \varepsilon_r} \left( \left[ J_0(k_1 a) \right]^2 + \left[ J_1(k_1 a) \right]^2 - \frac{2 J_0(k_1 a) J_1(k_1 a)}{k_1 a} \right)$$  [47]

$$P_s^0 = Re \left( \frac{\pi \omega_0^2 k_0^2 \beta}{2 \omega \varepsilon_0} \left( d^2 \left[ (H_0^{(1)}(k_2 d))^2 + [H_1^{(1)}(k_2 d)]^2 \right] - \frac{2H_0^{(1)}(k_2 d) H_1^{(1)}(k_2 d)}{k_2 d} \right) \right)$$

$$- a^2 \left( [H_0^{(1)}(k_2 a)]^2 + [H_1^{(1)}(k_2 a)]^2 - \frac{2H_0^{(1)}(k_2 a) H_1^{(1)}(k_2 a)}{k_2 a} \right)$$

---

**Table 4**

Values of $k_1$, $k_2$, and for the rod excited in $H E_{11}$ mode = 3.2 cm.

<table>
<thead>
<tr>
<th>$2a/\lambda_0$</th>
<th>$k_1$</th>
<th>$-jk_2$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>2.46</td>
<td>0.43</td>
<td>2.30</td>
</tr>
<tr>
<td>0.40</td>
<td>2.42</td>
<td>0.54</td>
<td>2.04</td>
</tr>
<tr>
<td>0.50</td>
<td>2.15</td>
<td>1.20</td>
<td>2.32</td>
</tr>
<tr>
<td>0.60</td>
<td>1.92</td>
<td>1.58</td>
<td>2.52</td>
</tr>
<tr>
<td>0.70</td>
<td>1.71</td>
<td>1.80</td>
<td>2.66</td>
</tr>
<tr>
<td>0.80</td>
<td>1.56</td>
<td>1.93</td>
<td>2.75</td>
</tr>
<tr>
<td>0.90</td>
<td>1.41</td>
<td>2.04</td>
<td>2.83</td>
</tr>
<tr>
<td>1.00</td>
<td>1.28</td>
<td>2.12</td>
<td>2.89</td>
</tr>
<tr>
<td>1.50</td>
<td>0.90</td>
<td>2.31</td>
<td>3.04</td>
</tr>
<tr>
<td>2.00</td>
<td>0.69</td>
<td>2.38</td>
<td>3.08</td>
</tr>
</tbody>
</table>
where $B$ and $D$ are related by the following relation

$$D = \frac{B}{\varepsilon_{r_1}} \left( \frac{k_1}{k_2} \right)^2 \left[ \frac{J_0(k_1a) H_0^{(1)}(k_2a)}{H_0^{(1)}(k_2a)} \right]$$

which is obtained by using the appropriate field components and boundary condition. The total power flow is therefore,

$$-\frac{\pi B^2 \beta k_1^2}{2 \omega \varepsilon_0 \varepsilon_{r_1}} \left[ u^2 \left( [J_0(k_1a)]^2 + [J_1(k_1a)]^2 - \frac{2 J_0(k_1a)}{k_1a} J_1(k_1a) \right) \right]$$

$$+ \left( \frac{1}{\varepsilon_{r_1}} \left( \frac{k_1^2}{k_2^2} \frac{J_0(k_1a)}{H_0^{(1)}(k_2a)} \right) \right) \left[ \frac{d^2 \left( [H_0^{(1)}(k_2d)]^2 + [H_1^{(1)}(k_2d)]^2 - \frac{2 H_0^{(1)}(k_2d)}{k_2d} H_1^{(1)}(k_2d) \right)}{k_2a} \right]$$

$$- \frac{2 H_0^{(1)}(k_2d)}{k_2a} \frac{H_1^{(1)}(k_2d)}{k_2a} \right]$$

Figures 18 and 19 respectively represent $W\%$ versus $\rho$ and constant percentage power contours.

21. CONCLUSIONS

The following interesting points emerge as a result of the above theoretical investigations.

(i) For the same value of $2a/\lambda_0$, and $\varepsilon_{r_1}$, more power is concentrated near the rod in the case of HE mode compared to the $E_0$ mode excitation.

(ii) The power flow inside the rod for HE mode excitation is more compared to that in the case of $E_0$ mode (12) excitation.

(iii) For the same value of $L$, $2a/\lambda_0$, $\varepsilon_{r_1}$, $\sigma\varepsilon$ and $\tan \delta$ the $Q$ factor of the resonator is greater in the case of $E_0$—mode than in the case of HE mode excitation. This is justified due to the fact that in the case of HE mode though the loss in the end plates due to finite $\sigma\varepsilon$ is less than $E_0$ mode due to less radial field spread, but the loss due to $\tan \delta$ in the case of HE—mode is comparatively much more than in the case of $E_0$—mode due to higher concentration of power inside the rod.

(iv) The nature of variation of the $Q$—factor with wavelength of excitation is practically the same in both the cases of excitation.

It may therefore be concluded that a dielectric rod excited in HE mode will act as a better surface wave guide than when excited in $E_0$ mode provided the dielectric rod is made of material having low value of loss tangent.
Fig. 14

Fig. 15
Theory of Open Resonator with an Axial Dielectric Rod

Fig. 16

Radius of the area in cm

$\epsilon_r = 2.56$

$d/\lambda_0 = 0.3$
Fig. 17
RADIUS OF THE ROD ' Q '

1. 1.27 cm
2. 1.4287 cm
3. 1.5875 cm
4. 1.9050 cm
5. 2.2225 cm
6. 2.54 cm

FIG. 18
22. ACKNOWLEDGEMENTS

The investigator-in-charge (S. K. Chatterjee) is indebted to the Dr. S. Dhawan, Director, Indian Institute of Science for providing necessary facilities for the work. He is also grateful to University Grants Commission, New Delhi for permission to conduct investigations on the project. He expresses his grateful thanks to Dr. James R. Wait, Monitor, for encouragement and to the U. S. Department of Commerce for providing PL-480 funds, and to N O O A., U.S.A.

23. REFERENCES

2. ______ and Zacharia, K. P. Radio and Electronic Engineer 1968, 36, 111.
9. ______ and Chatterjee, R. Ibid. 1968. 50, 345.
10. ______ and et. al. Ibid. 1971, 53, 63.
12. Shankara K. N., Chatterjee S. K., under publication.