SOME THEORETICAL INVESTIGATIONS ON DIELECTRIC ROD WAVEGUIDE

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ABSTRACT

The propagation characteristics of the HE_{11}-mode launched on a circular cylindrical dielectric rod wave guide, such as the radial propagation constant, the axial propagation constant and the guide wavelength at 3.2 cms, wavelength have been studied theoretically as a function of the diameter and the dielectric constant of the rod. The electric field distributions and the power flow as a function of the diameter and the dielectric constant of the rod have also been calculated for both inside and outside the rod.

INTRODUCTION

Several authors have investigated the propagation characteristics of electromagnetic waves launched on dielectric rods, planes, conducting plane with dielectric coating, corrugated plane and cylindrical conducting structures etc. The object of this paper is to present a theoretical study of the propagation characteristics of the HE_{11}-mode launched on a circular cylindrical dielectric rod, with particular reference to the variation of

(i) the radial propagation constants \( k_1 \) and \( k_2 \) inside and outside the dielectric rod respectively as a function of the diameter \( d \) and the relative dielectric constant \( \varepsilon_r \) of the rod at a wavelength of \( \lambda_0 = 3.2 \) cms.

(ii) the axial propagation constant \( \gamma \) as a function of \( d \) and \( \varepsilon_r \)

(iii) the guide wavelength \( \gamma \) as a function of \( d \) and \( \varepsilon_r \).

The electric field distributions \( (E_r, E_\theta, E_z) \) inside and outside the rod of dielectric constant \( \varepsilon_r = 2.6 \) and of different diameters have also been calculated. The power flow inside the rod as a percentage of the total power flow has also been calculated as a function of \( d \) and \( \varepsilon_r \) and some of the results are presented graphically.

Field Components: The field components inside and outside the circular cylindrical dielectric guide of radius \( r \) are the following:

Inside the dielectric, \( r \leq r \) (Medium 1)

\[
E_{\rho_1} = -B \left[ \frac{1}{\rho} J_1 (k_1 \rho) + \frac{b}{B} \frac{\gamma_1 k_1}{\iota \omega \varepsilon_1} J'_1 (k_1 \rho) \right] \sin \phi \cdot e^{-\gamma_1 \rho}
\]
Graphical solution of the characteristic equation [7] for $d/\lambda_0=0.8$, $\varepsilon_2=2.6$ and $\lambda_0=3.2$ cms.
where \( k_1 \) and \( k_2 \) are the radial propagation constants, \( \gamma_1 \) and \( \gamma_2 \) are the axial constants, \( \mu_1 \) and \( \mu_2 \) are the permeabilities of media 1 and 2 respectively, \( \varepsilon_1 \) and \( \varepsilon_2 \) are the dielectric constants of the two media, \( b, c, B \) and \( C \) are constants to be determined by the boundary conditions. The Bessel function \( J_1(x) \) and the Hankel function \( H^{(1)}_1(x) \) have been used for the field components inside
and outside the guide in order to ensure the regularity of the field on the axis of the guide and to satisfy the condition at infinity respectively. The time variation \( \exp(j\omega t) \) has been omitted from eqn. [2] for convenience.

**FIG. II**

Variation of the radial propagation constants \( k_1 \) and \( k_2 \) with \( d/\lambda_0 \) \((\lambda_0 = 3.2 \text{ cms})\)

(As \( k_2 \) is imaginary \(-i k_2\) has been plotted)

**Boundary condition:** The following boundary conditions at \( \rho = r \) are to be satisfied:

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FIG. III

Variation of the radial propagation constant $k_1$ and $k_3$ with $\bar{\varepsilon}_1$ ($d/\lambda_0=0.8, \lambda_0=3.2\text{ cms}$).
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\( E_{z1} = E_{z2}, \quad E_{\phi 1} = E_{\phi 2} \) \hspace{1cm} (a)

\( H_{z1} = H_{z2}, \quad H_{\phi 1} = H_{\phi 2} \) \hspace{1cm} (b)

It is also necessary, for a single hybrid mode to exist, that

\[ \gamma_1 = \gamma_2 = \gamma = \sqrt{(k_1^2 - \omega^2 \mu_1 \varepsilon_1)} = \sqrt{(k_2^2 - \omega^2 \mu_2 \varepsilon_2)} \] \hspace{1cm} (4)

**FIG. IV**

Variation of the axial propagation constant \( \gamma \), with \( d/\lambda_o \) for \( \lambda_o = 3.2 \) cm.
**Characteristic Equation**: From the equations [1], [2] and [3], the following equations are obtained:

\[
B k_1 J'_1 (k_1 r) + \frac{b}{r} \cdot \frac{\gamma}{i \omega \varepsilon_1} J_1 (k_1 r) = C k_2 H^{(1)'}_1 (k_2 r) + \frac{c}{r} \cdot \frac{\gamma}{i \omega \varepsilon_2} H^{(1)'}_1 (k_2 r)
\]

\[
b \frac{k_1^2}{i \omega \varepsilon_1} J_1 (k_1 r) = c \frac{k_2^2}{i \omega \varepsilon_2} H^{(1)}_1 (k_2 r)
\]  \[5\]

\[
\frac{B}{r} \cdot \frac{\gamma}{i \omega \mu_1} J_1 (k_1 r) + b k_1 J'_1 (k_1 r) = C \frac{\gamma}{i \omega \mu_2} H^{(1)}_1 (k_2 r) + C k_2 H^{(1)'}_1 (k_2 r)
\]

\[
B \frac{k_1^2}{i \omega \mu_1} J_1 (k_1 r) = C \frac{k_2^2}{i \omega \mu_2} H^{(1)}_1 (k_2 r)
\]  \[6\]

**FIG. V**

Variation of the axial propagation constant $\gamma$, with $\bar{\varepsilon}_1$, for $d/\lambda_b=0.8$ and $\lambda_b=3.2$ cms.
The equations [5] and [6] have a nontrivial solution if the determinant of the coefficients of the constants \( b, B, c \) and \( C \) vanishes. Expansion of this determinant leads to the following characteristic transcendental equation.

\[
\left[ \frac{1}{x_1} \frac{J'_1(x_1)}{J_1(x_1)} - \frac{1}{x_2} \frac{H_{11}^{(1)}(x_2)}{H_{11}^{(1)}(x_2)} \right] \left[ \frac{\varepsilon_1}{x_1} \frac{J'_1(x_1)}{J_1(x_1)} - \frac{1}{x_2} \frac{H_{11}^{(1)}(x_2)}{H_{11}^{(1)}(x_2)} \right] = \frac{(x_1^2 - x_2^2)(x_1^2 - \varepsilon_1 x_2^2)}{x_1^2 x_2^2}
\]

FIG. VI

Variation of \( \lambda_2/\lambda_0 \) with \( d/\lambda_0 \) for the \( HE_{11} \)-mode (\( \lambda_2 = 3.2 \) cms).
where
\[ x_1 = k_1 r; \quad x_2 = k_2 r \] [8]
and \( x_1 \) and \( x_2 \) are also related by the following equation:
\[ x_1^2 + \left( \frac{x_2}{i} \right)^2 = \left( \frac{\pi d}{\lambda_0} \right)^2 (\bar{\varepsilon}_1 - 1) \] [9]

**Solution of the Characteristic equation:** The characteristic equation [7] has been solved in the following way:

Let
\[ Y_1 = \frac{1}{x_1} \left( \frac{J_1'(x_1)}{J_1(x_1)} - \frac{1}{x_2} \frac{H_1^{(1)'}(x_2)}{H_1^{(1)}(x_2)} \right) \left[ \frac{\bar{\varepsilon}_1}{x_1} \frac{J_1'(x_1)}{J_1(x_1)} - \frac{1}{x_2} \frac{H_1^{(1)'}(x_2)}{H_1^{(1)}(x_2)} \right] \] [10]

and
\[ Y_2 = \frac{(x_1^2 - x_2^2)(x_1^2 - \bar{\varepsilon}_1 x_2^2)}{x_1^2 x_2^4} \] [11]

\( Y_1 \) and \( Y_2 \) are expressed as a function of \( x_1 \) only with the help of the equation [9]. The values of \( (x_2/i) \) are calculated corresponding to some arbitrary values of \( x_1 \) for a particular value of \( d/\lambda_0 \) and \( \bar{\varepsilon}_1 \). \( Y_1 \) and \( Y_2 \) are then plotted as a function of \( x_1 \). The intersection of \( Y_1 \) and \( Y_2 \) gives the root \( x_1 \) and hence \( x_2 \) of the equation [7]. The procedure is repeated for different values of \( d/\lambda_0 \) and \( \bar{\varepsilon}_1 \). As an illustration the graphical solution for \( d/\lambda_0 = 0.8 \) and \( \bar{\varepsilon}_1 = 2.6 \) is presented in Fig. I.

**Radial and Axial propagation constants:** The radial propagation constants \( k_1 \) and \( k_2 \) are calculated from equation [8]. They have been plotted as a function of \( d/\lambda_0 \) for \( \bar{\varepsilon}_1 = 2.6 \) in Fig. II, and a function of \( \bar{\varepsilon}_1 \) for \( d/\lambda_0 = 0.8 \) in Fig. III. The axial propagation constant \( \gamma \) is calculated from equation [4] and its variation with \( d/\lambda_0 \) and \( \bar{\varepsilon}_1 \) is shown in Fig. IV and V respectively. It is observed that \( k_1 \) decreases and \( k_2 \) and \( \gamma \) increase with increasing values of \( d \) and \( \bar{\varepsilon}_1 \) and this gives rise to the reduction of the radial field spread.

**Guide Wavelength:** The guide wavelength \( \lambda_g \) is obtained from equations [4] and [9] and is as follows:
\[ \frac{\lambda_g}{\lambda_0} = \left[ \bar{\varepsilon}_1 - \left( \frac{x_1}{\pi d/\lambda_0} \right)^2 \right]^{-1} \] [12]
The values of $\lambda_{g}/\lambda_{0}$ as computed from the above expression are plotted in Fig. VI as a function of $d/\lambda_{0}$ (from 0.2 to 1.8) and $\bar{\varepsilon}_{1} = 2.6$ and as a function of $\bar{\varepsilon}_{1}$ (from 1 to 10) for $d/\lambda_{0} = 0.8$ in Fig. VII. It is observed that $\lambda_{g}$ decreases with an increase in $\bar{\varepsilon}_{1}$ as is expected. The decrease of $\lambda_{g}$ with an increase in the diameter of the rod may be due to the increase of $\gamma$, as an increase in $\gamma (= i\beta)$ for a lossless waveguide reduces the phase velocity of the propagating wave. The reduction in phase velocity is responsible for the decrease of $\lambda_{g}$ with increase of $d$.

Field Configuration:—The constants $b$, $B$, $c$ and $C$ as obtained from equations [5] and [6] are given by the following expressions:—

$$\frac{b}{B} = \frac{\gamma \varepsilon_{1}}{i \omega \mu_{1}} \frac{x_{1}x_{2}}{x_{1}^{2} - x_{2}^{2}} \left[ \frac{\varepsilon_{1}}{x_{1}} J'_{1}(x_{1}) - \frac{\varepsilon_{2}}{x_{2}} H^{(1)}_{1}(x_{2}) \right]^{-1}$$

[13]
The constants are evaluated for different values of \( d/\lambda_0 \) and \( \varepsilon_1 \). The electric field components \( (E_r, E_\theta \text{ and } E_z) \) inside and outside the rod are then evaluated for different values of \( d/\lambda_0 \) and some of the results are presented graphically in Figures VIII, IX and X. The values of the field components have been normalized.

\[
\begin{align*}
\frac{c}{B} &= \frac{x_1^2}{x_2^2} \frac{\mu_2}{\mu_1} \frac{J_1(x_1)}{H^{(1)}_1(x_2)} \\
\frac{c}{b} &= \frac{x_1^2}{x_2^2} \frac{\varepsilon_2}{\varepsilon_1} \frac{J_1(x_1)}{H^{(1)}_1(x_2)}
\end{align*}
\]

and

\[
\frac{c}{C} = \frac{1}{i \omega \mu_2} \frac{x_1^2 - x_2^2}{x_1^2 x_2^2} \left[ \frac{\varepsilon_1}{x_1} J_1(x_1) - \frac{\varepsilon_2}{x_2} H^{(1)}_1(x_2) \right]^{-1}
\]

**Fig. VIII**
Variation of the radial component, \( E_\rho \), of the electric field with the radial distance, \( \rho \).

\((\varepsilon_1 = 2.6, \lambda_0 = 3.2 \text{ cm})\).

\(E_\rho\) has been normalized with respect to its value \( E'_\rho \) on the surface of the rod.
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with respect to their values at the surface of the rod. The decrease of $k_1$ and
the increase of $k_2$ and $\gamma$ with the increase of $d$ are responsible for the reduction
of the field spread in the radial direction.

**Power Flow:** The power flow, $w_1$, along the $z$-direction inside the rod per
unit area is given by

$$w_1 = \frac{1}{2} \left[ E_{\rho_1} H_{\Phi_1}^* - E_{\Phi_1} H_{\rho_1}^* \right]$$

[17]

The power flow inside the rod is

$$W_i = \frac{1}{2} \int_\rho_0^r \int_\phi_0^{2\pi} \left[ E_{\rho_1} H_{\Phi_1}^* - E_{\Phi_1} H_{\rho_1}^* \right] \rho \, d\rho \, d\phi$$

[18]

which yields

$$W_i = B^2 \left[ \frac{b}{B} \frac{1}{\mu_1} k_1 \left( 1 - \frac{\gamma^2}{\omega^2 \mu_1 \epsilon_1} \right) \int_\rho_0^r J_0(k_1 \rho) J_1(k_1 \rho) \, d\rho \right.$$ 

$$- \frac{\pi \gamma}{i \omega} \frac{k_1}{1 + \frac{b^2}{B^2} \frac{1}{\epsilon_1}} \int_\rho_0^r J_0(k_1 \rho) J_1(k_1 \rho) \, d\rho$$

$$+ \frac{\pi \gamma}{i \omega} \frac{1}{1 + \frac{b^2}{B^2} \frac{1}{\epsilon_1}} \int_\rho_0^r \frac{1}{\rho} \left[ J_1(k_1 \rho) \right]^2 d\rho$$

$$- \frac{b}{B} \pi \left( 1 - \frac{\gamma^2}{\omega^2 \mu_1 \epsilon_1} \right) \int_\rho_0^r \frac{1}{\rho} \left[ J_1(k_1 \rho) \right]^2 d\rho$$

$$+ \frac{\pi \gamma k_1^2}{2 i \omega} \left( \frac{1}{\mu_1} + \frac{b^2}{B^2} \frac{1}{\epsilon_1} \right) \int_\rho_0^r \rho \left[ J_0(k_1 \rho) \right]^2 d\rho \right]$$

[19]

The power flow outside the rod is

$$W_0 = \frac{1}{2} \int_\rho_0^\infty \int_\phi_0^{2\pi} \left[ E_{\rho_2} H_{\Phi_2}^* - E_{\Phi_2} H_{\rho_2}^* \right] \rho \, d\rho \, d\phi$$

[20]
Variation of the azimuthal component $E_\phi$, of the electric field with the radial distance, $\rho$.

$(\varepsilon_1 = 2.6, \lambda_0 = 3.2\text{ cms}).$

($E_\phi$ has been normalized with respect to its value $E_0^\prime$ on the surface of the rod).

Variation of the axial component, $E_z$, of the electric field with radial distance $\rho$.

$(\varepsilon_1 = 2.6, \lambda_0 = 3.2\text{ cms}).$
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FIG. XI
Percentage power flow inside the waveguide vs. \( d/\lambda_0 \) \((\epsilon_r=2.6, \lambda_0=3.2\) cms\).

FIG. XII
Percentage power flow inside the waveguide vs. \( \epsilon_r \) \((d/\lambda_0=0.8, \lambda_0=3.2\) cms\).
which yields

\[
W_0 = C^2 \left[ \frac{c}{C} \pi k_2 \left( 1 - \frac{\gamma^2}{\omega^2 \mu_2 \varepsilon_2} \right) \int_{\rho=r}^{\infty} H_0^{(1)}(k_2 \rho) H_1^{(1)}(k_2 \rho) \, d\rho \right. \\
- \frac{\pi \gamma k_2}{i \omega} \left( \frac{1}{\mu_2} + \frac{c^2}{C^2} \cdot \frac{1}{\varepsilon_2} \right) \int_{\rho=r}^{\infty} H_0^{(1)}(k_2 \rho) H_1^{(1)}(k_2 \rho) \, d\rho \right. \\
+ \frac{\pi \gamma}{i \omega} \left( \frac{1}{\mu_2} + \frac{c^2}{C^2} \cdot \frac{1}{\varepsilon_2} \right) \int_{\rho=r}^{\infty} \frac{1}{\rho} \left[ H_1^{(1)}(k_2 \rho) \right]^2 \, d\rho \\
- \frac{c}{C} \pi \left( 1 - \frac{\gamma^2}{\omega^2 \mu_2 \varepsilon_2} \right) \int_{\rho=r}^{\infty} \frac{1}{\rho} \left[ H_1^{(1)}(k_2 \rho) \right]^2 \, d\rho \\
+ \frac{\pi \gamma k_2}{2 i \omega} \left( \frac{1}{\mu_2} + \frac{c^2}{C^2} \cdot \frac{1}{\varepsilon_2} \right) \int_{\rho=r}^{\infty} \rho \left[ H_0^{(1)}(k_2 \rho) \right]^2 \, d\rho \right] 
\]

The expressions for \( W_i \) and \( W_0 \) are evaluated by numerical integration for different values of \( d/\lambda_0 \) and \( \varepsilon_1 \). From these values of \( W_i \) and \( W_0 \), \( W_i \) is expressed as a percentage of the total power \( W_T \) flowing in the rod and some of the results are presented in Fig. XI and XII.

\[
\frac{W_i}{W_T} = \frac{(W_i/W_0)}{1 + (W_i/W_0)}
\]

It is observed that the power flow is more and more concentrated inside the guide as the dielectric constant or the diameter of the rod is increased. This is justified by the fact that the radial field spread is reduced as \( d \) or \( \varepsilon_1 \) increases.

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