TRIANGULAR DECOMPOSITIONS AND THEIR APPLICATIONS IN SOLVING EQUATIONS

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ABSTRACT

Discussed here are all possible triangular decompositions and their role to solve linear equations. Also presented are the computational formulae for five of these triangular decompositions to provide illustrations regarding their computational use.

1. Triangular Decompositions

We define a triangular matrix as a square matrix having zero elements above (or below) the left diagonal (or right diagonal). The following four sets of triangular matrices are considered for the decomposition of a square matrix and the solution of linear equations. They are:

\[
\begin{align*}
L_l, P_l & = \begin{pmatrix} 1 \end{pmatrix}, & L_r, P_r & = \begin{pmatrix} 1 \end{pmatrix} \\
R_l, U_l & = \begin{pmatrix} 1 \end{pmatrix}, & R_r, U_r & = \begin{pmatrix} 1 \end{pmatrix}
\end{align*}
\]

\[L_l, P_l\] are called lower triangular matrices of left diagonal \( L_r, P_r \) are called lower triangular matrices of right diagonal \( R_l, U_l \) are upper triangular matrices of left diagonal \( R_r, U_r \) are upper triangular matrices of right diagonal. 1, 2, 3, 4, 5, 6, 7, 8 are numerical symbols to denote \( L_l (= l_{ij}) \), \( P_l (= p_{ij}) \), \( L_r (= l'_{ij}) \), \( P_r (= p'_{ij}) \), \( R_l (= r_{ij}) \), \( U_l (= u_{ij}) \), \( R_r (= r'_{ij}) \), \( U_r (= u'_{ij}) \), respectively. The two triangular matrices \( L_l \) and \( P_l \) have the identical form but different elements. Similar is the case with \( L_r, P_r \); \( R_l, U_l \); \( R_r, U_r \).

The following chart (chart 1) gives us all possible triangular decompositions \([i.e., \text{decomposition of a square matrix into two triangular matrices (product form)}]\) including invalid ones.
Triangular Decompositions and their Applications in Solving Equations

### Chart 1

<table>
<thead>
<tr>
<th>x12</th>
<th>x23</th>
<th>.34</th>
<th>x45</th>
<th>x56</th>
<th>x67</th>
<th>.78</th>
</tr>
</thead>
<tbody>
<tr>
<td>x21</td>
<td>.32</td>
<td>.43</td>
<td>.54</td>
<td>x65</td>
<td>.76</td>
<td>.87</td>
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<tr>
<td>x13</td>
<td>x24</td>
<td>x35</td>
<td>x46</td>
<td>x57</td>
<td>x68</td>
<td></td>
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<tr>
<td>.31</td>
<td>.42</td>
<td>.53</td>
<td>.64</td>
<td>.75</td>
<td>.86</td>
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</tr>
<tr>
<td>x14</td>
<td>.25</td>
<td>x36</td>
<td>x47</td>
<td>x58</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.41</td>
<td>.52</td>
<td>.63</td>
<td>x74</td>
<td>.85</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.15</td>
<td>.26</td>
<td>x37</td>
<td>x48</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.51</td>
<td>.62</td>
<td>x73</td>
<td>x84</td>
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<td>.16</td>
<td>.27</td>
<td>x38</td>
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<tr>
<td>.61</td>
<td>x72</td>
<td>x83</td>
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<tr>
<td>.17</td>
<td>.28</td>
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<tr>
<td>x71</td>
<td>x82</td>
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<td>.18</td>
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<tr>
<td>x81</td>
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</tbody>
</table>

'x' indicates invalid triangular decompositions as these decompositions produce triangular forms in the product and consequently not valid to represent (or replace) a square matrix. There are totally 28 ($=6+4+6+4+2$) invalid triangular decompositions out of 56 ($=8^2-8$). '.' indicates the valid triangular decompositions. There are totally 28 ($=56-28$) valid triangular decompositions that can be used to replace a square matrix. All these 28 valid decompositions are not different as can be seen from the following chart (chart 2).

### Chart 2

<table>
<thead>
<tr>
<th>15=25≡16≡26</th>
<th>51≡52≡61≡62</th>
<th>17≡27≡18≡28</th>
</tr>
</thead>
<tbody>
<tr>
<td>31≡41≡32≡42</td>
<td>75≡85≡76≡86</td>
<td>53≡54≡63≡64</td>
</tr>
<tr>
<td>34≡43</td>
<td>78≡87</td>
<td></td>
</tr>
</tbody>
</table>

'≡' indicates 'is identical in nature'. Thus there are eight different triangular decompositions that can represent a square matrix uniquely.

**Failures:** When a matrix possesses one or more vanishing leading minors on left diagonal $L_iR_i$ (i.e., 15) algorithm collapses. The $R_iL_i$ (i.e., 51) algorithm fails for one or more trailing minors that are zero on the left diagonal. $L_iR_i$ (i.e., 17) and $L_iL_i$ (i.e., 31) algorithms, on the other hand, blows up for some leading vanishing minors on the right diagonal. The $R_iR_i$
(i.e., 75) decomposition meets the aforesaid ill-fate for some vanishing trailing minors on the right diagonal. Finally we put forth the fact that all the eight triangular decompositions collapse for a coefficient matrix $A$ possessing vanishing leading and trailing minors on both left and right diagonals even if the coefficient matrix $A$ may be non-singular. The following matrix, for instance,

$$
\begin{pmatrix}
2 & 1 & -1 & -3 \\
4 & 2 & 2 & 6 \\
5 & 6 & -5 & -2 \\
5 & 6 & -10 & -4
\end{pmatrix}
$$

is not decomposable by any triangular decomposition though it is highly non-singular.

We are now confronted with two situations. Should we go on trying triangular decompositions one after the other or should we restrict ourselves to only one of the eight decompositions using the row (or column) interchanging technique? In case we possess the idea of the leading and trailing minors, we can go ahead with the suitable decomposition. Otherwise, we follow the latter process.

2. Computational Algorithms

Derived here are the simple explicit computational recurrence relations for obtaining $L_p, R_p, L_r$ and $R_r$ and their inverses along with the solution vector $\tilde{X}$ from the aforesaid five useful decompositions making use of matrix operation rules. The summation sign used below has the conventional meaning, i.e., the summation has to be taken as zero if its upper bound is less than the lower bound. The '=>' sign in any recurrence relation has the identical meaning as that in Fortran. The 'equivalent' sign, on the other hand, carries the meaning 'equivalent' all throughout.

We store the matrix $A$ and the column vector $\vec{b}$ as below:

$$
A = (a_{ij}), \quad i=1, 2, \ldots, n; \quad j=1, 2, \ldots, n
$$

and

$$
\vec{b} = (b_1, b_2, \ldots, b_n)'
$$

where (') indicates transpose. The sequence of subscripts $i$ and $j$ (to represent matrix elements) in

$$
\begin{align*}
'i=1, 2, \ldots, n; \quad j=1, 2, \ldots, n'
\end{align*}
$$

indicate that $i$ is to take 1 (fixed) first and $j$ is to be varied from 1 to $n$ at an interval of 1. Then $i=2$ (fixed) and $j=1, 2, 3, \ldots, n$. Next $i=3$ (fixed),
\( j = 1, 2, \ldots, n \) and so on. At every change of subscripts, we determine different elements. A brief note for the mode of change of subscripts, however, is added in some cases for easy and quick access to the recurrence relations.

If none of the leading minors on the left diagonal are zeros, then
\[
A = L_i R_i, \quad l_{ii} = 1 \ \forall \ i
\]

\( j = 1, 2, \ldots, n \)

\[
R_i: \quad a_{ij} = a_{ij} - \sum_{p=1}^{i-1} a_{ip} a_{pj}
\]

\( i = 1, 2, \ldots, j \)

\[
L_i: \quad a_{ij} = (a_{ij} - \sum_{p=1}^{j-1} a_{ip} a_{pj})/a_{ii}
\]

\( i = j+1, j+2, \ldots, n \)

Letting \( j = 1 \), we obtain \( a_{11} (\equiv r_{11}) \) from the first recurrence relation of (1.1) and then \( a_{21}, a_{31}, \ldots, a_{n1} (\equiv l_{31}, l_{41}, \ldots, l_{n1}) \) from the second one. For \( j = 2 \), \( a_{22}, a_{32}, \ldots, a_{n2} (\equiv l_{22}, l_{32}, \ldots, l_{n2}) \) are determined from the first relation and \( a_{32}, a_{42}, \ldots, a_{n2} (\equiv l_{32}, l_{42}, \ldots, l_{n2}) \) from the second relation and so on. Lastly, for \( j = n \) we find \( a_{1n}, a_{2n}, \ldots, a_{nn} (\equiv r_{1n}, r_{2n}, \ldots, r_{nn}) \) from the first relation.

\[
\text{Det } A = \prod_{i=1}^{n} a_{ii}
\]

In \( L_i R_i \mathbf{x} = \mathbf{b} \ i \ e., \) in \( R_i L_i^{-1} \mathbf{x} = \mathbf{c} \), we have the elements of \( L_i^{-1} \) matrix:

\[
L_i^{-1}: \quad a_{ij} = -a_{ij} - \sum_{p=1}^{i-1} a_{ip} a_{pj}
\]

\( j = 1, 2, \ldots, n-1; \quad i = j+1, j+2, \ldots, n \)

The elements of the column vector \( \mathbf{c} \) are:

\[
\mathbf{c}: \quad b_i = b_i + \sum_{p=1}^{i-1} a_{ip} b_p
\]

\( i = n, n-1, \ldots, 2 \) (\( b_1 \) remains unchanged)

The solution vector \( \mathbf{x} \) is:

\[
\mathbf{x}: \quad b_i = (b_i - \sum_{p=1}^{i+1} a_{ip} b_p)/a_{ii}
\]

\( i = n, n-1, \ldots, 1 \)
where $x = (x_1, x_2, \ldots, x_n)' = (b_1, b_2, \ldots, b_n)'$.

When none of the trailing minors on the left diagonal are zeros, then

$$A = R_j L_j, \quad r_{ii} = 1 \quad \forall \quad i$$

$$j = n, \quad n - 1, \ldots, 1$$

$L_j$: \[a_{ij} = a_{ij} - \sum_{p=1}^{n} a_{pj} a_{ip} \]

$$i = n, \quad n - 1, \ldots, j$$

$R_j$: \[a_{ij} = (a_{ij} - \sum_{p=1}^{n} a_{jp} a_{ip}) / a_{ij} \]

$$i = j - 1, \quad j - 2, \ldots, 1$$

The elements of $L_j$ and $R_j$ matrices have the identical subscripts with those of $A$.

$$\text{Det} \ A = \prod_{i=1}^{n} a_{ii}$$

In $R_j L_j \xrightarrow{R_j^{-1}} x = b$ i.e., in $L_j x = R_j^{-1} b = c$, we have the elements of $R_j^{-1}$ matrix:

$$R_j^{-1}: \quad a_{ij} = a_{ij} - \sum_{p=1}^{n} a_{ip} a_{pj}$$

$$j = n, \quad n - 1, \ldots, 2; \quad i = j - 1, \quad j - 2, \ldots, 1$$

The elements of the column vector $c$ are:

$$c: \quad b_i = b_i' + \sum_{p=i+1}^{n} a_{ip} b_p$$

$$i = 1, 2, \ldots, n - 1$$

The solution vector $x$ is:

$$x: \quad b_i = (b_j - \sum_{p=1}^{i-1} a_{ip} b_p) / a_{ii}$$

$$i = 1, 2, \ldots, n$$

where $x = (x_1, x_2, \ldots, x_n)' = (b_1, b_2, \ldots, b_n)'$

If none of the leading minors are vanishing on the right diagonal, then

$$A = L_j R_j, \quad l_{ii} = 1 \quad \forall \quad i$$
\[ \begin{align*}
\text{For } i = 1, 2, \ldots, n \\
L_i: \quad a_{i, n-j+1} &= (a_{i, n-j+1} - \sum_{p=1}^{j-1} a_{i, n-p+1} a_{p, n-j+1}) / a_{i, n-j+1} \\
j = 1, 2, \ldots, i - 1
\end{align*} \]

\[ \begin{align*}
R_r: \quad a_{ij} &= a_{ij} - \sum_{p=1}^{n-i} a_{n, n-p+1} a_{pj} \\
j = 1, 2, \ldots, (n-i+1)
\end{align*} \]

For \( i = 1 \), we find \( a_{11}, a_{12}, \ldots, a_{1n} (\equiv r'_{11}, r'_{12}, \ldots, r'_{1n}) \) from the second recurrence relation of [3.1]. For \( i = 2 \), we obtain \( a_{2n} (\equiv l_{21}) \) from the first relation and then we find \( a_{21}, a_{22}, \ldots, a_{2, n-1} (\equiv r'_{21}, r'_{22}, \ldots, r'_{2, n-1}) \) from the second relation. For \( i = 3 \), we determine \( a_{3n}, a_{3, n-1} (\equiv l_{31}, l_{32}) \) from the first relation and \( a_{31}, a_{32}, \ldots, a_{3, n-2} (\equiv r'_{31}, r'_{32}, \ldots, r'_{3, n-2}) \) from the second relation, and so on. Finally, for \( i = n \), we find \( a_{nn}, a_{n, n-1}, \ldots, a_{n2} (\equiv l_{n1}, l_{n2}, \ldots, l_{n, n-1}) \) from the first relation and \( a_{n1} (\equiv r'_{n1}) \) from the second relation.

\[ \text{Det } A = (-1)^{[n/2]} \prod_{i=1}^{n} a_{i, n-i+1} \] [3.2]

where \([n/2]\) is the integral part of \( n/2 \).

In \( L \), \( r \rightarrow x = b \) i.e., in \( R_r \rightarrow x = L_i^{-1} b = c \), we have the elements of \( L_i^{-1} \) matrix:

\[ \begin{align*}
L_i^{-1}: \quad a_{i, n-j+1} &= -a_{i, n-j+1} - \sum_{p=1}^{j-1} a_{i, n-p+1} a_{p, n-j+1} \\
j = 1, 2, \ldots, i, \quad n - 1; \quad i = j + 1, j + 2, \ldots, n
\end{align*} \]

The elements of the column vector \( c \) are:

\[ \begin{align*}
\rightarrow c: \quad b_j &= b_j + \sum_{p=1}^{j-1} a_{j, n-p+1} b_p \\
j = n, n-1, \ldots, 2
\end{align*} \]

where \( c = (c_1, c_2, \ldots, c_n)' \equiv (b_1, b_2, \ldots, b_n)' \)

The solution vector \( x \) is:

\[ \begin{align*}
\rightarrow x: \quad b_i &= (b_i - \sum_{p=1}^{n-i} a_{ip} b_{n-p+1}) / a_{i, n-i+1} \\
i = n, n-1, \ldots, 1
\end{align*} \]

\( \rightarrow \) denotes direction.
where \( x = (x_1, x_2, \ldots, x_n)' \equiv (b_n, b_{n-1}, \ldots, b_1)' \)

When none of the leading minors vanishes on the right diagonal,

\[
A = L_r \cdot L_f, \quad l'_{i, n-i+1} = 1 \quad \forall \; i
\]

\[
L_f:\quad a_{n-i+1, j} = a_{n-i+1, j} - \sum_{p=1}^{i-1} a_{n-p+1, j} a_{n-i+1, p}
\]

\[
L_r:\quad c_i = (a_{i-1} - \sum_{p=i}^{n} a_{n-p+1, j} a_{n-j+1, i})
\]

For \( j = n \), we find \( a_{mn} (l_{mn}) \) from the first recurrence relation and \( a_{2n}, a_{3n}, \ldots, a_{nn} (l'_{2n}, l'_{3n}, \ldots, l'_{nn}) \) from the second relation. For \( j = n-1 \), we calculate \( a_1, a_2, a_3, \ldots, a_{n-1} (\equiv l_{n-1}, l_{n-1}, \ldots, l'_{n-1}) \) from the first relation and \( a_2, a_3, a_4, \ldots, a_n, a_{n-1} \) (\( \equiv l'_{n-1}, l'_{n-1}, \ldots, l'_{n-1} \)) from the second relation. For \( j = n-2 \), we get \( a_1, a_2, a_3, a_4, \ldots, a_{n-2} (\equiv l_{n-2}, l_{n-2}, l_{n-2}, l_{n-2}) \) from the first relation and \( a_2, a_3, a_4, \ldots, a_n, a_{n-2} \) (\( \equiv l'_{n-2}, l'_{n-2}, l'_{n-2}, \ldots, l'_{n-2} \)) from the second relation and so on. Lastly, for \( j = 1 \), we find \( a_{11}, a_{21}, \ldots, a_{n1} (\equiv l_{11}, l_{11}, \ldots, l_{11}) \) from the first relation.

\[
\text{Det } A = (-1)^{[n/2]} \prod_{i=1}^{n} a_{n-i+1, i}
\]

In \( L_r \cdot L_f x = b \), \( \text{i.e., } L_f x = L_r^{-1} b = c \), we have the elements of \( L_r^{-1} \) matrix:

\[
L_r^{-1}: \quad a_{ij} = -a_{ij} - \sum_{p=n-i+2}^{j-1} a_{ip} a_{n-p+1, j}
\]

\( j = n, n-1, \ldots, 2 \); \( i = n-j+2, n-j+3, \ldots, n \)

The elements of the column vector \( c \) are:

\[
c: \quad b_j = b_j + \sum_{p=n-j+2}^{n} a_{jp} b_{n-p+1}
\]

\( j = n, n-1, \ldots, 2 \)

\( b_1 \) remains unchanged
where \( \mathbf{c} = (c_1, c_2, \ldots, c_n)' \equiv (b_n, b_{n-1}, \ldots, b_1)' \)

The solution vector \( \mathbf{x} \) is:

\[
\mathbf{x} = (h_i \leftarrow \frac{\sum_{p=1}^{n-i} a_{pi} b_{n-p+1}}{a_{ni}}, n-i+1)
\]

\( i = n, n-1, \ldots, 1 \)

where \( \mathbf{x}' = (x_1, x_2, \ldots, x_n)' \equiv (b_n, b_{n-1}, \ldots, b_1)' \)

When none of the trailing minors vanishes on the right diagonal,

\[
A = R_r R_l, \quad r_{ii} = 1 \quad \forall \ i
\]

\( i = n, n-1, \ldots, 1 \)

\[
R_r:\ \ a_{ij} = a_{ij} \leftarrow \sum_{p=1}^{j-1} \frac{a_{ip} a_{n-p+1}, j}{a_{ni}, n-i+1}
\]

\( j = 1, 2, \ldots, n-i+1 \)

\[
R_l:\ \ a_{ij} = a_{ij} \leftarrow \sum_{p=1}^{n-i} \frac{a_{ip} a_{n-p+1}, j}{a_{ni}, n-i+1}
\]

\( j = n-i+2, n-i+3, \ldots, n \)

For \( i = n \), we find \( a_{n1} (\equiv r'_{11}) \) from the first recurrence relation and \( a_{n2}, a_{n3}, \ldots, a_{nn} (\equiv r'_{12}, r'_{13}, \ldots, r'_{1n}) \) from the second relation. For \( i = n-1 \), \( a_{n-1, 1}, a_{n-1, 2} (\equiv r'_{n-1, 1}, r'_{n-1, 2}) \) are obtained from the first relation and \( a_{n-1, 3}, a_{n-1, 4}, \ldots, a_{n-1, n'} (\equiv r'_{23'}, r'_{24'}, \ldots, r'_{2n'}) \) from the second relation.

For \( i = n-2 \), we calculate \( a_{n-2, 1}, a_{n-2, 2} a_{n-2, 3} (\equiv r'_{n-2, 1}, r'_{n-2, 2}, r'_{n-2, 3}) \) from the first relation and \( a_{n-2, 4}, a_{n-2, 5}, \ldots, a_{n-2, n} (\equiv r'_{24}, r'_{25}, \ldots, r'_{2n}) \) from the second relation, and so on. Lastly, for \( i = 1 \), we find \( a_{11}, a_{12}, \ldots, a_{1n} (\equiv r'_{11}, r'_{12}, \ldots, r'_{1n}) \) from the first relation.

\[
\text{Det } A = (-1)^{n+1} \prod_{i=1}^{n} a_{i, n-i+1} [5.2]
\]

\( \text{In } R_r R_l x = b, \text{ i.e., in } R_l x = R_r^{-1} b = c \), we have the elements of \( R_r^{-1} \) matrix:

\( R_r^{-1}: \ a_{ij} = 1/a_{ij} \text{ if } i+1 > n-j+1 \)

otherwise,

\[
a_{ij} = \left( -\sum_{p=1}^{n-i+1} a_{pj} a_{i, n-p+1}\right)a_{i, n-i+1}
\]

\( j = 1, 2, \ldots, n; \ i = n-j+1, n-j+1, \ldots, 1 \)
The elements of the column vector $\mathbf{c}$ are:

$$\mathbf{c} : \quad c_j = \sum_{p=1}^{n} a_{jp} \cdot h_{p+1} \quad j = 1, 2, \ldots, n$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_n)' \equiv (h_n, h_{n-1}, \ldots, h_1)'$

The solution vector $\mathbf{x}$ is:

$$\mathbf{x} : \quad x_i = b_i - \sum_{p=i+1}^{n} a_{ip} \cdot h_{n-p+1} \quad i = 1, 2, \ldots, n$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)' \equiv (h_n, h_{n-1}, \ldots, h_1)'$

3. Remarks

To encounter the unforeseen cases where one or more leading (or trailing) minors vanish on the left (or the right) diagonal, we reserve a shortage of $(n+1)$ locations for the current row (or column) of the augmented matrix $(\mathbf{A}, \mathbf{b})$. As soon as an element on the left (or right) diagonal becomes zero, we swing to the interchange of rows (or columns) from the current row (or column) onwards. It is interesting to note that any one of the eight decompositions (that cannot be restarted because of the gradual destruction of the original matrix in the computer memory) can proceed to obtain the solution vector $\mathbf{x}$ for any nonsingular system.

Gaussian algorithms (Gauss, Doolittle, Crout, Cholesky and Banchiewicz) are, however, basically the same as our $L_R$ algorithm and consequently have to be performed with row (or column) interchanging technique for vanishing leading minors on the left diagonal.

References